

ON THE STRONG LAW OF LARGE NUMBERS FOR  
SUMS OF RANDOM ELEMENTS IN BANACH SPACES

By

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A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL  
OF THE UNIVERSITY OF FLORIDA IN PARTIAL FULFILLMENT  
OF THE REQUIREMENTS FOR THE DEGREE OF  
DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

2001

## ACKNOWLEDGMENTS

I would like to thank my advisor, Andrew Rosalsky, for his inspirational attitude and understanding. His direction and motivation have made this work possible. I would also like to thank the members of my dissertation committee, Dr. Malay Ghosh, Dr. Ramon Littell, Dr. James Hobert, and Dr. Irene Hueter. I thank Dr. Ron Randles for his support and encouragement before and throughout my study at the University of Florida.

I thank my parents for always believing that I could accomplish my goals, and for their constant love and support. I thank my husband, Emory, for his love, encouragement and understanding.

I give my thanks to all of the teachers that inspired and motivated my desire to learn, with special thanks to Rita Cater, Ron Goolsby, James Bentley, and Thomas Polaski. I wish to give special thanks to Mrs. Anne Close for her many years of support, encouragement and friendship.

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Abstract of Dissertation Presented to the Graduate School  
of the University of Florida in Partial Fulfillment of the  
Requirements for the Degree of Doctor of Philosophy

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May 2001

Chairman: Andrew Rosalsky  
Major Department: Statistics

For a sequence of random elements  $\{V_n, n \geq 1\}$  in a real separable Banach space  $\mathcal{X}$ , sufficient conditions are provided for the strong law of large numbers  $\sum_{i=1}^n (V_i - c_i)/b_n \rightarrow 0$  almost certainly to hold where  $\{c_n, n \geq 1\}$  and  $\{b_n > 0, n \geq 1\}$  are suitable sequences of centering elements in  $\mathcal{X}$  and norming constants, respectively.

In the case of independent random elements, separate necessary conditions are also provided for a strong law of large numbers. The necessity result extends a real line result of Martikainen to a Banach space setting. The sufficiency results in the case of independent summands are new even when  $\mathcal{X}$  is the real line and are divided into two categories. The first assumes that  $\mathcal{X}$  is of Rademacher type  $p$  ( $1 \leq p \leq 2$ ). The result is general enough to include as special cases a strong law of Adler, Rosalsky, and Taylor for sums of independent and identically distributed random elements and a strong law of Heyde for sums of independent (real-valued) random variables. The second imposes no conditions on the underlying Banach space; instead, it assumes

that the sequence of random elements is compactly uniformly integrable. The result includes as special cases a strong law of Adler, Rosalsky, and Taylor for compactly uniformly integrable sequences and a strong law of Taylor and Wei for uniformly tight sequences.

Strong laws are also provided where no conditions are imposed on the joint distributions of the random elements or on the underlying Banach space, and these results are new even when the Banach space is the real line.

Illustrative examples are provided which compare the results or which show how the results improve upon or are different from other results in the literature. Examples are also provided which show that the results are sharp.

## CHAPTER 1

### INTRODUCTION

The history and literature of investigation on *laws of large numbers* are vast and rich, as this concept is crucial in probability and statistical theory and in their application. There is nothing more fundamental to the very *foundation* of statistical science than the laws of large numbers. It is the laws of large numbers which elucidate the notion that probability is a limiting relative frequency and hence the laws of large numbers provide justification for Kolmogorov's (1933) axiomatic theory of probability being a physically realistic subject. Indeed, the laws of large numbers provide a rigorous formulation and justification for the notion that "the sample mean approaches the population mean as the sample size approaches infinity" (i.e., the sample mean is a consistent estimator of the population mean). It is the area of consistency wherein the laws of large numbers have many applications.

The first theorem on the law of large numbers, due to Bernoulli in the late 1600's, was the *weak law of large numbers* (WLLN) for Bernoulli trials which states that if  $S_n$  is the number of successes observed in  $n$  independent identical trials with success probability  $p$  in each trial, then  $S_n/n \rightarrow p$  in probability as  $n \rightarrow \infty$ . Bernoulli's result was touted by Kolmogorov in 1986 as the beginning of probability proper (see Bingham (1989)). After Lebesgue's work on measure theory in the early 1900's, the (first) *strong law of large numbers* (SLLN) could be formulated. The first of such SLLNs is credited to Borel (1909) and is the extension of Bernoulli's result from convergence in probability to convergence with probability one (or almost certain (a.c.) convergence). It is interesting to observe that there was over a two hundred

year gap between this SLLN and its earlier WLLN counterpart. Cantelli (1917) is credited with the first SLLN regarding the a.c. convergence of the sample mean to the population mean (see Seneta (1992)).

The laws of large numbers provide the consistency of many estimators including numerous common statistics (such as, of course, the sample mean), as well as for estimates found by Monte Carlo simulation. Many of these concepts and applications can be extended. Interesting applications of SLLNs also occur in physics and computer science. The field of ergodic theory, which lies at the interface of probability theory and statistical physics, has as its foundation the SLLN type result provided by the Birkoff-Khintchine-von Neumann pointwise ergodic theorem (see, e.g., Breiman (1968, Chapter 6) or Stout (1974, Section 3.5)). This result gives conditions for an “ensemble average” to be estimated by a “time average.” A SLLN was derived by Wehr (1997) with applications to random resistor networks and the durability of composite fibers. Other applications of the SLLN to resistance can be found in Essoh and Bellissard (1989). Another interesting and “non-statistical” application occurs in the field of number theory in the investigation of *normal* numbers  $x \in [0, 1]$  (see Révész (1968, pp. 151-157)).

The primary objective of the current work is to investigate conditions under which the SLLN for independent Banach space valued random elements obtains, although some results will be presented for which the assumption of independence is not needed. Before discussion of this objective some notation and definitions must be introduced.

Let  $\mathcal{X}$  be a linear space over  $\mathbb{R}$ ; that is,  $\mathcal{X}$  is a vector space over  $\mathbb{R}$ . Let  $\|\cdot\| : \mathcal{X} \rightarrow [0, \infty)$  be a function satisfying the following three properties:

- (i) For  $v \in \mathcal{X}$ ,  $\|v\| = 0$  if and only if  $v = 0$ .

- (ii)  $\|av\| = |a|\|v\|$  for all  $v \in \mathcal{X}$  and  $a \in \mathbb{R}$ .
- (iii)  $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$  for all  $v_1, v_2 \in \mathcal{X}$ .

The function  $\|\cdot\|$  is called a *norm* on  $\mathcal{X}$ , and  $\mathcal{X}$  is then said to be a real *normed linear space* (with norm  $\|\cdot\|$ ).

A sequence  $\{v_n, n \geq 1\}$  in  $\mathcal{X}$  is said to *converge* to  $v \in \mathcal{X}$  if  $\lim_{n \rightarrow \infty} \|v_n - v\| = 0$ . A real normed linear space is said to be *complete* if every Cauchy sequence in  $\mathcal{X}$  converges to a member of  $\mathcal{X}$ . (A Cauchy sequence in  $\mathcal{X}$  is, of course, a sequence  $\{v_n, n \geq 1\}$  in  $\mathcal{X}$  such that

$$\lim_{\substack{n \rightarrow \infty}} \|v_n - v_m\| = 0.)$$

A real normed linear space  $\mathcal{X}$  which is complete is said to be a real *Banach space*. A real Banach space is said to be *separable* if it contains a countable dense subset.

Let  $(\Omega, \mathcal{F}, P)$  be a probability space, let  $\mathcal{X}$  be a real separable Banach space with norm  $\|\cdot\|$ , and let  $\mathcal{X}$  be equipped with its Borel  $\sigma$ -algebra  $\mathcal{B}$ ; that is,  $\mathcal{B}$  is the  $\sigma$ -algebra generated by the class of open subsets of  $\mathcal{X}$  determined by  $\|\cdot\|$  via the metric  $d(v_1, v_2) = \|v_1 - v_2\|$ ,  $v_1, v_2 \in \mathcal{X}$ . A *random element*  $V$  in  $\mathcal{X}$  is an  $\mathcal{F}$ -measurable transformation from  $\Omega$  to the measurable space  $(\mathcal{X}, \mathcal{B})$ . The *expected value* or *mean* of  $V$ , denoted  $EV$ , is defined to be the *Pettis integral* provided it exists; that is,  $V$  has expected value  $EV$  in  $\mathcal{X}$  if  $L(EV) = E(L(V))$  for every  $L$  in  $\mathcal{X}^*$  where  $\mathcal{X}^*$  denotes the (*dual*) space of all continuous linear functionals on  $\mathcal{X}$ . We recall that a *linear functional* is a function  $L : \mathcal{X} \rightarrow \mathbb{R}$  satisfying  $L(av + bw) = aL(v) + bL(w)$  for all  $v, w \in \mathcal{X}$  and all  $a, b \in \mathbb{R}$ . Of course, a linear functional  $L$  is *continuous* if whenever  $\{v_n, n \geq 1\}$  is a sequence in  $\mathcal{X}$  with  $v_n \rightarrow v \in \mathcal{X}$  we have  $L(v_n) \rightarrow L(v)$ . A sufficient condition for  $EV$  to exist is that  $E\|V\| < \infty$  (see, e.g., Taylor (1978, p. 40)). A complete characterization of when the Pettis integral exists was provided by Brooks

(1969). See Hille and Phillips (1957, pp. 76-85) for further discussion and details regarding the properties of the Pettis integral.

An example illustrating expectation in Banach spaces follows: Let  $1 \leq p < \infty$  and  $\mathcal{X} = \mathcal{L}_p(\mathbb{R})$  where, as usual, this denotes the class of Lebesgue measurable functions  $f: \mathbb{R} \rightarrow \mathbb{R}$  such that  $\int_{\mathbb{R}} |f(x)|^p dx < \infty$  where the norm is defined by  $\|f\| = \left( \int_{\mathbb{R}} |f(x)|^p dx \right)^{1/p}$ . This class is a Banach space according to the famous Riesz-Fischer theorem (see, for example, Royden (1988, p. 125)). For fixed  $f_0 \in \mathcal{X}$  and integrable random variables  $X$  and  $Y$ , define the random element  $V = Xf_0 + Y$ . We claim that  $EV = E(X)f_0 + E(Y)$ . To verify this, note that for any  $L \in \mathcal{X}^*$ , we have  $L(E(X)f_0 + E(Y)) = E(X)L(f_0) + E(Y)L(1) = E(XL(f_0)) + E(YL(1)) = E(XL(f_0) + YL(1)) = E(L(Xf_0 + Y)) = E(L(V))$  and thus the candidate  $EV = E(X)f_0 + E(Y)$  satisfies the defining property  $L(EV) = E(L(V))$ . We remark that the dual space  $\mathcal{X}^*$  can be completely characterized for this example. For every  $g \in \mathcal{L}_q(\mathbb{R})$  where  $q = \frac{p}{p-1}$  ( $1 < q \leq \infty$ ), the functional  $L$  defined by

$$L(f) = \int_{\mathbb{R}} f(x)g(x) dx \quad (f \in \mathcal{L}_p) \quad (1.1)$$

is in  $\mathcal{X}^*$  (see, e.g., Royden (1988, p. 131)). Conversely, the famous Riesz Representation Theorem (see, e.g., Royden (1988, p. 132)) asserts that every  $L \in \mathcal{X}^*$  is of the form (1.1) for some function  $g \in \mathcal{L}_q(\mathbb{R})$  where  $q = \frac{p}{p-1}$  ( $1 < q \leq \infty$ ). The function  $g$  is unique up to sets of Lebesgue measure 0.

Let  $\{Y_n, n \geq 1\}$  be a symmetric *Bernoulli sequence*; that is,  $\{Y_n, n \geq 1\}$  is a sequence of independent and identically distributed (i.i.d.) random variables with  $P\{Y_1 = 1\} = P\{Y_1 = -1\} = 1/2$ . Let  $\mathcal{X}^\infty = \mathcal{X} \times \mathcal{X} \times \mathcal{X} \times \dots$  and define  $\mathcal{C}(\mathcal{X}) = \{(v_1, v_2, \dots) \in \mathcal{X}^\infty : \sum_{n=1}^{\infty} Y_n v_n \text{ converges in probability}\}$ . Let  $1 \leq p \leq 2$ . Then  $\mathcal{X}$  is said to be of *Rademacher type p* if there exists a constant  $0 < C < \infty$

such that

$$E \left\| \sum_{n=1}^{\infty} Y_n v_n \right\|^p \leq C \sum_{n=1}^{\infty} \|v_n\|^p \text{ for all } (v_1, v_2, \dots) \in \mathcal{C}(\mathcal{X}).$$

It should be pointed out that the condition that the series  $\sum_{n=1}^{\infty} Y_n v_n$  converges in probability is equivalent to the condition that the series  $\sum_{n=1}^{\infty} Y_n v_n$  converges a.c. Indeed, Itô and Nisio (1968) proved that a series of *independent* random elements converges in probability if and only if it converges a.c. thereby extending the celebrated theorem of Lévy (see, e.g., Chow and Teicher (1997, p. 72)) from the real line to Banach spaces.

Hoffmann-Jørgensen and Pisier (1976) proved for  $1 \leq p \leq 2$  that a real separable Banach space is of Rademacher type  $p$  if and only if there exists a constant  $0 < C < \infty$  such that

$$E \left\| \sum_{i=1}^n V_i \right\|^p \leq C \sum_{i=1}^n E \|V_i\|^p \quad (1.2)$$

for every finite collection  $\{V_1, \dots, V_n\}$  of independent random elements with  $EV_i = 0$ ,  $1 \leq i \leq n$ . Other characterizations of a Banach space being of Rademacher type  $p$  are provided by Woyczyński (1978).

If a real separable Banach space is of Rademacher type  $p$  for some  $1 < p \leq 2$ , then it is of Rademacher type  $q$  for all  $1 \leq q < p$ . Every real separable Banach space is of Rademacher type (at least) 1 while the  $\mathcal{L}_p$ -spaces and  $\ell_p$ -spaces are of Rademacher type  $\min\{2, p\}$  for  $p > 1$ . (See Woyczyński (1978, pp. 343-353) for details and a thorough discussion.) Every real separable Hilbert space and real separable finite-dimensional Banach space is of Rademacher type 2 (see Pisier (1986)). In particular, the real line  $\mathbb{R}$  is of Rademacher type 2.

A sequence of random elements  $\{V_n, n \geq 1\}$  is said to be *compactly uniformly integrable* if for every  $\varepsilon > 0$ , there exists a compact subset  $K_\varepsilon$  of  $\mathcal{X}$  such that

$$\sup_{n \geq 1} E \|V_n I(V_n \notin K_\varepsilon)\| \leq \varepsilon.$$

A sequence of random elements being compactly uniformly integrable is the natural extension of a sequence of (real-valued) random variables being uniformly integrable in that the two definitions are equivalent when the Banach space is the real line.

A sequence of random elements  $\{V_n, n \geq 1\}$  is said to be *uniformly tight* if for every  $\varepsilon > 0$ , there exists a compact subset  $K_\varepsilon$  of  $\mathcal{X}$  such that  $\sup_{n \geq 1} P\{V_n \notin K_\varepsilon\} \leq \varepsilon$ . It was shown by Daffer and Taylor (1982) that compact uniform integrability implies uniform tightness. Cuesta and Matrán (1988) remarked that if  $\{V_n, n \geq 1\}$  is compactly uniformly integrable, then  $\{\|V_n\|, n \geq 1\}$  is uniformly integrable. Cuesta and Matrán (1988) also remarked that  $\{\|V_n\|, n \geq 1\}$  being uniformly integrable together with  $\{V_n, n \geq 1\}$  being uniformly tight imply that  $\{V_n, n \geq 1\}$  is compactly uniformly integrable. Consequently,  $\{V_n, n \geq 1\}$  is compactly uniformly integrable if and only if  $\{V_n, n \geq 1\}$  is uniformly tight and  $\{\|V_n\|, n \geq 1\}$  is uniformly integrable. See Wang and Bhaskara Rao (1987) and Cuesta and Matrán (1988) for further discussion concerning the relationship between compact uniform integrability and uniform tightness.

Let  $\{V_n, n \geq 1\}$  be a sequence of random elements and, as usual, their partial sums will be denoted by

$$S_n = \sum_{i=1}^n V_i, \quad n \geq 1.$$

We say that the sequence of random elements  $\{V_n, n \geq 1\}$  obeys the *strong law of large numbers* (SLLN) with centering elements  $\{C_n, n \geq 1\}$  and norming constants  $\{b_n > 0, n \geq 1\}$  if

$$\frac{S_n - C_n}{b_n} \rightarrow 0 \text{ a.c.} \quad (1.3)$$

Here 0 denotes of course the zero element of  $\mathcal{X}$  and the centering elements  $\{C_n, n \geq 1\}$  are (nonrandom) members of  $\mathcal{X}$ . The SLLN in (1.3) can of course be expressed as

$$\frac{1}{b_n} \sum_{i=1}^n (V_i - c_i) \rightarrow 0 \text{ a.c.}$$

where  $c_n = C_n - C_{n-1}$ ,  $n \geq 1$  taking  $C_0 = 0$  ( $\sum_{i=1}^n c_i = C_n$ ,  $n \geq 1$ ).

The history of the SLLN problem for random elements in a real separable Banach space dates back to the pioneering work of Mourier (1953) (see Laha and Rohatgi (1979, p. 452) or Taylor (1978, p. 72)) which established an analogue of the classical Kolmogorov SLLN. Specifically, Mourier showed that if  $\{V_n, n \geq 1\}$  are i.i.d. random elements in a real separable Banach space and if  $EV_1$  exists, then  $\sum_{i=1}^n (V_i - EV_1)/n \rightarrow 0$  a.c. (For alternative proofs of Mourier's SLLN, see Cuesta and Matrán (1986) and Adler, Rosalsky, and Taylor (1989).) In the case of i.i.d. (real-valued) random variables  $\{X_n, n \geq 1\}$  with  $E|X_1| < \infty$ , the Kolmogorov SLLN (which has norming constants  $b_n = n$ ,  $n \geq 1$ ) was generalized by Marcinkiewicz and Zygmund (1937) when  $E|X_1|^r < \infty$  for some  $0 < r < 2$  and the norming constants are  $b_n = n^{1/r}$ ,  $n \geq 1$ . Following these results, Feller (1946) in a famous article developed a result for general  $b_n$ . In the case of random elements, Woyczyński (1980) developed a Marcinkiewicz-Zygmund type SLLN. (For some related results, see Wang and

Bhaskara Rao (1987).) A random element version of Feller's SLLN which contains Woyczyński's result was obtained by Adler, Rosalsky, and Taylor (1989).

It should be noted that many of the Banach space SLLN results which are obtained in the current work are new results even in the (real-valued) random variable case. However, some corollaries or special cases of some of the random element results (and random variable results) are well known thereby showing that the current work is an extension of previously established results. Note that we are using the terminology *random element* when the underlying Banach space  $\mathcal{X}$  is general whereas we are using the terminology *random variable* when the underlying Banach space  $\mathcal{X}$  is the real line  $\mathbb{R}$ . (Hence every random variable is a random element with  $\mathcal{X} = \mathbb{R}$ .)

Banach space SLLNs have applications to many statistical problems. In many instances, it is useful to regard a continuous time stochastic process as a random element in a function space. This notion has inspired the study of convergence of random elements (see Taylor (1978)). Likewise, the natural extension of random variable results to random vector results in multivariate analysis also led to the study of random elements. The consideration of a manufacturing process as a random element (stochastic process) provides an example of the use of the laws of large numbers in the estimation of the drift over time for a continuous time stochastic process where under various conditions the SLLN can provide the consistency of the estimate (see Taylor (1978, pp. 186-190)). Taylor (1978, pp. 190-201) also discusses applications of the laws of large numbers in Banach spaces to decision theory, quality control, as well as estimation problems such as Monte Carlo processes and M-estimation. An interesting example is the extension of Monte Carlo methods to estimating a solution  $\theta$ , where  $\theta$  is an element of a Banach space  $\mathcal{X}$  using a sequence of estimators  $\{V_n, n \geq 1\}$  where each  $V_n$  is a random element in  $\mathcal{X}$  with  $EV_n = \theta$  (see Taylor (1978, pp. 196-197)). Reeds (1978) proved the strong consistency of jackknifed maximum

likelihood estimates utilizing a Banach space SLLN. Such SLLNs have also been used to prove the strong consistency of least squares estimates of unknown parameters in multiple regression models, autoregressive models, and time series models (see Lai, Robbins, and Wei (1979), Chen, Lai, and Wei (1981), and Lai and Wei (1983)).

Despite the fact that the previously mentioned applications of SLLNs represent extremely valuable contributions to modern statistical practice, our objective in the current work does not include application of the SLLNs presented herein. Rather, our objective is to increase our *understanding* of the strong limiting behavior of random sums with Banach space valued or real valued summands.

Often SLLNs are derived from results for a.c. convergence of series of random elements via the Kronecker lemma. This lemma states that if  $\{v_n, n \geq 1\}$  is a sequence in  $\mathcal{X}$ , if  $\{b_n, n \geq 1\}$  is a real sequence with  $0 < b_n \uparrow \infty$ , and if  $\sum_{n=1}^{\infty} (v_n/b_n)$  converges, then  $(\sum_{i=1}^n v_i)/b_n \rightarrow 0$ . Thus if  $\{V_n, n \geq 1\}$  is a sequence of random elements and  $\{c_n, n \geq 1\}$  is a sequence in  $\mathcal{X}$  and if  $\sum_{n=1}^{\infty} \frac{V_n - c_n}{b_n}$  converges a.c., then by the Kronecker lemma the SLLN  $\frac{1}{b_n} \sum_{i=1}^n (V_i - c_i) \rightarrow 0$  a.c. obtains. Kolmogorov completely resolved the question of necessary and sufficient conditions for almost certain convergence of series of independent random variables with his celebrated three series criterion (see, e.g., Chow and Teicher (1997, p. 117)). Thus in the random variable case many SLLNs can be obtained by applying the three series criterion (or some corollary thereof) followed by the Kronecker lemma. But it must be realized that the Kronecker lemma approach yields a sufficient but not necessary condition for the SLLN. This approach was apparently first used by Rademacher (1922).

It should be pointed out that for *independent* summands there is no loss of generality in only considering norming sequences  $\{b_n, n \geq 1\}$  with  $0 < b_n \uparrow \infty$  (as opposed to  $0 < b_n \rightarrow \infty$ ). Indeed, in the case of independent random variables  $\{X_n, n \geq 1\}$ , Martikainen (1979) proved that  $\{X_n, n \geq 1\}$  obeys the SLLN with

norming constants  $\{b_n, n \geq 1\}$  if and only if  $\{X_n, n \geq 1\}$  obeys the (apparently sharper) SLLN with norming constants  $\{b_n^* \equiv \inf_{j \geq n} b_j, n \geq 1\}$ . In the next chapter a similar result will be proven in the more general case of random elements. Moreover, note that for  $b_n > 0$ ,  $b_n \rightarrow \infty$  if and only if  $b_n^* = \inf_{j \geq n} b_j \uparrow \infty$ . Working with monotone  $\{b_n, n \geq 1\}$  has the added advantage that the Kronecker lemma is then applicable. Thus only monotone  $\{b_n, n \geq 1\}$  will be considered when the summands are independent.

Throughout, it proves convenient to define  $a \vee b = \max\{a, b\}$ ,  $a \wedge b = \min\{a, b\}$  ( $a, b \in \mathbb{R}$ ) and  $\log x = \log_e(e \vee x)$ ,  $x > 0$  where  $\log_e$  denotes the logarithm to the base  $e$ . The symbol  $[x]$  denotes the greatest integer less than or equal to  $x$ ,  $x \geq 0$ . Moreover, the symbol  $C$  denotes a generic constant ( $0 < C < \infty$ ) which is not necessarily the same one in each appearance. Thus, for example, an expression such as  $(2C^2 + 1)(n + n^{\frac{1}{2}}) \leq Cn$  is valid in the current context.

CHAPTER 2  
STRONG LAWS OF LARGE NUMBERS FOR SUMS OF RANDOM ELEMENTS

**2.1 Introduction**

At the origin of the current investigation concerning the SLLN problem is the following well-known result of Heyde (1968) establishing a SLLN for sums of independent random variables.

**Proposition 2.1.1.** (Heyde (1968)) *Let  $\{X_n, n \geq 1\}$  be a sequence of independent random variables. If  $\{b_n, n \geq 1\}$  is a sequence of positive constants with  $b_n \uparrow \infty$  and*

$$\sum_{n=1}^{\infty} E\left(\frac{X_n^2}{X_n^2 + b_n^2}\right) < \infty, \quad (2.1)$$

*then the SLLN*

$$\frac{1}{b_n} \sum_{i=1}^n \left( X_i - E(X_i I(|X_i| \leq b_i)) \right) \rightarrow 0 \text{ a.c.} \quad (2.2)$$

*obtains.*

**Remark 2.1.1.** As was shown by Heyde (1968), the condition (2.1) is equivalent to the pair of conditions

$$\sum_{n=1}^{\infty} P\{|X_n| > b_n\} < \infty, \quad (2.3)$$

and

$$\sum_{n=1}^{\infty} \frac{1}{b_n^2} E(X_n^2 I(|X_n| \leq b_n)) < \infty. \quad (2.4)$$

The following example shows that the SLLN (2.2) can hold even though (2.1) fails.

**Example 2.1.1.** Let  $\{X_n, n \geq 1\}$  be a sequence of independent random variables and  $\{\alpha_n, n \geq 1\}$  be a sequence of constants with  $1 \leq \alpha_n \uparrow \infty$ ,  $\sum_{n=1}^{\infty} \frac{1}{\alpha_n^2} = \infty$ , and

$$P\left\{X_n = \frac{2^n}{\alpha_n}\right\} = P\left\{X_n = \frac{-2^n}{\alpha_n}\right\} = \frac{1}{2}, \quad n \geq 1.$$

Let  $b_n = 2^n$ ,  $n \geq 1$ , and note that (2.4) does not hold since

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{b_n^2} E(X_n^2 I(|X_n| \leq b_n)) &= \sum_{n=1}^{\infty} \frac{1}{2^{2n}} E(X_n^2 I(|X_n| \leq 2^n)) \\ &= \sum_{n=1}^{\infty} \frac{1}{2^{2n}} \left( \frac{2^{2n}}{\alpha_n^2} \right) \\ &= \sum_{n=1}^{\infty} \frac{1}{\alpha_n^2} \\ &= \infty. \end{aligned} \quad (2.5)$$

Hence Proposition 2.1.1 cannot be applied. However, using only first principles, we have for all  $n \geq 2$ , with probability 1

$$\frac{1}{2^n} \left| \sum_{i=1}^n \left( X_i - E(X_i I(|X_i| \leq 2^i)) \right) \right| \leq \frac{1}{2^n} \sum_{i=1}^n |X_i|$$

$$\begin{aligned}
&= \frac{1}{2^n} \sum_{i=1}^n \frac{2^i}{\alpha_i} \\
&= \frac{1}{2^n} \sum_{i=1}^{\lceil n/2 \rceil} \frac{2^i}{\alpha_i} + \frac{1}{2^n} \sum_{i=\lceil n/2 \rceil + 1}^n \frac{2^i}{\alpha_i} \\
&\leq \frac{1}{2^n} \sum_{i=1}^{\lceil n/2 \rceil} 2^i + \frac{1}{2^n \alpha_{\lceil n/2 \rceil + 1}} \sum_{i=\lceil n/2 \rceil}^n 2^i \\
&\leq \frac{2^{\lceil n/2 \rceil + 1}}{2^n} + \frac{2^{n+1}}{2^n \alpha_{\lceil n/2 \rceil + 1}} \\
&= \frac{2}{2^{n/2}} + \frac{2}{\alpha_{\lceil n/2 \rceil + 1}} \\
&\rightarrow 0.
\end{aligned}$$

Thus the SLLN

$$\frac{1}{2^n} \sum_{i=1}^n \left( X_i - E(X_i I(|X_i| \leq 2^i)) \right) \rightarrow 0 \text{ a.c.} \quad (2.6)$$

obtains.

Even though (2.6) was derived using only first principles, it illustrates some limitations of Proposition 2.1.1. This suggests the need to investigate the conditions of the proposition in order to understand the above limitations. Three questions that arise are the following:

- Can (2.4) be weakened to yield the conclusion (2.2)?
- Do any new conditions need to be added to a weaker version of (2.4)?
- Under what condition can the truncated expectation in (2.2) be replaced by the actual expectation of  $X_i$  (assuming the  $X_i$  are in  $\mathcal{L}_1$ )?

In this chapter these questions will be answered in the more general context of the case of a sequence of independent random elements. The conclusion of Example 2.1.1 will be shown to follow immediately from a more general form of Proposition 2.1.1 which will be obtained in Section 2.3. See Theorem 2.3.1, Examples 2.1.1 and 2.4.1, and Remarks 3.3.2 and 3.3.3. Theorem 2.3.1 thus extends Proposition 2.1.1 in two directions, namely:

- Theorem 2.3.1 pertains to a sequence of independent Banach space valued random elements rather than to a sequence of independent random variables as does Proposition 2.1.1.
- Theorem 2.3.1, when specialized to the random variable case, pertains to a wider class of sequences of random variables than does Proposition 2.1.1.

However, before we proceed with establishing Theorem 2.3.1, we provide in the following section necessary conditions for a SLLN with independent Banach space valued random element summands.

## 2.2 Necessary Conditions for a SLLN

The first theorem, Theorem 2.2.1, provides *necessary* conditions for independent Banach space valued random elements to obey a SLLN. The random elements  $\{V_n, n \geq 1\}$  are assumed to be *symmetric*; that is,  $V_n$  and  $-V_n$  are identically distributed for all  $n \geq 1$ . Of course it is assumed that not all of the  $\{V_n, n \geq 1\}$  are degenerate. (A random element  $V$  is said to be *degenerate* if for every  $B \in \mathcal{B}$ ,  $P\{V \in B\} \in \{0, 1\}$ .) Thus it is assumed for some  $n \geq 1$  and some  $B_n \in \mathcal{B}$  that  $P\{V_n \in B_n\} > 0$  and  $P\{V_n \in B_n^c\} > 0$ .

Conclusions (2.8) and (2.9) of Theorem 2.2.1 were, in effect, proved by Martikainen (1979) in the random variable case. The argument presented below for (2.9)

is entirely different and more straightforward than that of Martikainen (1979). (See Kesten (1972) and Rosalsky and Teicher (1981) for some related results concerning law of the iterated logarithm type behavior for the random variable case.)

The following technical lemma is used in the proof of (2.8) of Theorem 2.2.1. For a random element  $V$ , let  $\sigma(V)$  denote the  $\sigma$ -algebra generated by  $V$ .

**Lemma 2.2.1.** *The sum of two independent random elements in a real separable Banach space  $\mathcal{X}$  (at least one of which is nondegenerate) is nondegenerate.*

**Proof.** Let  $V_1$  and  $V_2$  be independent random elements where  $V_1$  is nondegenerate and suppose that  $V_1 + V_2 = v$  a.c. for some  $v \in \mathcal{X}$ . Then  $V_2$  is nondegenerate. Since  $V_1$  is independent of  $V_2$  and since  $V_1 = v - V_2$  a.c., we have  $\sigma(v - V_2)$  is independent of  $\sigma(V_2)$ . But  $\sigma(v - V_2) = \sigma(V_2)$  and hence  $\sigma(V_2)$  is independent of itself. Then for every  $A \in \sigma(V_2)$ ,  $P\{A\} = P\{AA\} = P\{A\} \cdot P\{A\} = (P\{A\})^2$  implying  $P\{A\}$  is either 0 or 1. This contradicts the fact that  $V_2$  is nondegenerate.  $\square$

**Theorem 2.2.1.** *Let  $S_n = \sum_{i=1}^n V_i, n \geq 1$  where  $\{V_n, n \geq 1\}$  is a sequence of independent symmetric random elements (not all degenerate) in a real separable Banach space. If*

$$\frac{S_n}{b_n} \rightarrow 0 \text{ a.c.} \quad (2.7)$$

for some sequence of positive constants  $\{b_n, n \geq 1\}$ , then

$$b_n \rightarrow \infty, \quad (2.8)$$

$$\frac{S_n}{b_n^*} \rightarrow 0 \text{ a.c.} \quad (2.9)$$

where  $0 < b_n^* \equiv \inf_{j \geq n} b_j \uparrow \infty$ , and

$$\sum_{n=1}^{\infty} P\{\|V_n\| > \varepsilon b_n\} < \infty \text{ for all } \varepsilon > 0. \quad (2.10)$$

**Remark 2.2.1.** (i) Theorem 2.2.1 establishes that condition (2.7) and the apparently stronger (in view of  $b_n^* \leq b_n$ ,  $n \geq 1$ ) assertion (2.9) are indeed equivalent.  
(ii) It should be noted that the symmetry hypothesis will not be used in the proof of (2.8).

**Proof of Theorem 2.2.1.** To prove (2.8), assume that it fails. Then there exist a constant  $0 < M < \infty$  and a subsequence  $n(k) \uparrow \infty$  such that  $b_{n(k)} \leq M$ ,  $k \geq 1$ . Then for arbitrary  $\varepsilon > 0$

$$\begin{aligned} P\{\|S_{n(k)}\| \geq \varepsilon \text{ i.o.}(k)\} &\leq P\left\{\frac{\|S_{n(k)}\|}{b_{n(k)}} \geq \frac{\varepsilon}{M} \text{ i.o.}(k)\right\} \\ &\leq P\left\{\frac{\|S_n\|}{b_n} \geq \frac{\varepsilon}{M} \text{ i.o.}(n)\right\} \\ &= 0 \quad (\text{by (2.7)}). \end{aligned}$$

Thus

$$S_{n(k)} \rightarrow 0 \text{ a.c.} \quad (2.11)$$

Choose  $k_0$  so that at least one  $V_i$  ( $1 \leq i \leq n(k_0)$ ) is nondegenerate. Now (2.11) can be rewritten as

$$\sum_{i=1}^{n(k_0)} V_i + \lim_{k \rightarrow \infty} \sum_{i=n(k_0)+1}^{n(k)} V_i = 0 \text{ a.c.} \quad (2.12)$$

By Lemma 2.2.1,  $\sum_{i=1}^{n(k_0)} V_i$  is nondegenerate. Then by the independence of  $\sum_{i=1}^{n(k_0)} V_i$  and  $\lim_{k \rightarrow \infty} \sum_{i=n(k_0)+1}^{n(k)} V_i$ , we have by again applying Lemma 2.2.1 that their sum must be nondegenerate which contradicts (2.12). Thus (2.8) holds.

Assertion (2.9) will now be proved. It is clear that  $b_n^* \uparrow \infty$  in view of (2.8). Now for each  $n \geq 1$ ,  $b_n^* = b_{j(n)}$  for some  $j(n) \geq n$ . Note at the outset that (2.7) ensures that  $S_n/b_n \xrightarrow{P} 0$  which, in turn, ensures that

$$\frac{S_{j(n)}}{b_{j(n)}} \xrightarrow{P} 0. \quad (2.13)$$

Now for arbitrary  $\varepsilon > 0$  and all large  $n$  and all  $k \geq n$  we have (interpreting  $\sum_{i=k+1}^k V_i$  as 0)

$$\begin{aligned} P\left\{\left\|\sum_{i=k+1}^{j(k)} V_i\right\| \leq \frac{\varepsilon}{2} b_{j(k)}\right\} &= 1 - P\left\{\left\|\sum_{i=k+1}^{j(k)} V_i\right\| > \frac{\varepsilon}{2} b_{j(k)}\right\} \\ &= 1 - P\left\{\|S_{j(k)} - S_k\| > \frac{\varepsilon}{2} b_{j(k)}\right\} \\ &\geq 1 - \left(P\left\{\|S_{j(k)}\| > \frac{\varepsilon}{4} b_{j(k)}\right\} + P\left\{\|S_k\| > \frac{\varepsilon}{4} b_{j(k)}\right\}\right) \\ &\geq 1 - \left(P\left\{\|S_{j(k)}\| > \frac{\varepsilon}{4} b_{j(k)}\right\} + P\left\{\max_{1 \leq k' \leq j(k)} \|S_{k'}\| > \frac{\varepsilon}{4} b_{j(k)}\right\}\right) \\ &\geq 1 - \left(P\left\{\|S_{j(k)}\| > \frac{\varepsilon}{4} b_{j(k)}\right\} + 2P\left\{\|S_{j(k)}\| > \frac{\varepsilon}{4} b_{j(k)}\right\}\right) \end{aligned}$$

(by the random element version of Lévy's inequality (see, e.g.,

Araujo and Giné (1980, p. 102)))

$$\begin{aligned} &= 1 - 3P\left\{\|S_{j(k)}\| > \frac{\varepsilon}{4} b_{j(k)}\right\} \\ &\geq \frac{1}{2} \quad \text{(by (2.13))}. \end{aligned} \quad (2.14)$$

Now for all  $k \geq n \geq 1$ , the event  $\left[ \left\| \sum_{i=k+1}^{j(k)} V_i \right\| \leq \frac{\varepsilon}{2} b_{j(k)} \right]$  and the class of events  $\{ \left\| S_r \right\| > \varepsilon b_{j(r)} \} : r = n, \dots, k \}$  are independent. Thus by the Lemma for Events (see, e.g., Loève (1977, p. 258)), for all large  $n$

$$\begin{aligned} P \left\{ \bigcup_{k=n}^{\infty} \left[ \left\| S_k \right\| > \varepsilon b_{j(k)} \right] \left[ \left\| \sum_{i=k+1}^{j(k)} V_i \right\| \leq \frac{\varepsilon}{2} b_{j(k)} \right] \right\} \\ \geq P \left\{ \bigcup_{k=n}^{\infty} \left[ \left\| S_k \right\| > \varepsilon b_{j(k)} \right] \right\} \cdot \inf_{k \geq n} P \left\{ \left\| \sum_{i=k+1}^{j(k)} V_i \right\| \leq \frac{\varepsilon}{2} b_{j(k)} \right\} \\ \geq \frac{1}{2} P \left\{ \bigcup_{k=n}^{\infty} \left[ \left\| S_k \right\| > \varepsilon b_{j(k)} \right] \right\} \quad (\text{by (2.14)}). \end{aligned} \quad (2.15)$$

Hence

$$\begin{aligned} P \left\{ \frac{\left\| S_n \right\|}{b_n^*} > \varepsilon \text{ i.o.}(n) \right\} &= \lim_{n \rightarrow \infty} P \left\{ \bigcup_{k=n}^{\infty} \left[ \left\| S_k \right\| > \varepsilon b_{j(k)} \right] \right\} \\ &\leq 2 \lim_{n \rightarrow \infty} P \left\{ \bigcup_{k=n}^{\infty} \left[ \left\| S_k \right\| > \varepsilon b_{j(k)} \right] \left[ \left\| \sum_{i=k+1}^{j(k)} V_i \right\| \leq \frac{\varepsilon}{2} b_{j(k)} \right] \right\} \quad (\text{by (2.15)}) \\ &\leq 2 \lim_{n \rightarrow \infty} P \left\{ \bigcup_{k=n}^{\infty} \left[ \left\| S_{j(k)} \right\| > \frac{\varepsilon}{2} b_{j(k)} \right] \right\} \\ &= 2P \left\{ \frac{\left\| S_{j(n)} \right\|}{b_{j(n)}} > \frac{\varepsilon}{2} \text{ i.o.}(n) \right\} \\ &\leq 2P \left\{ \frac{\left\| S_n \right\|}{b_n} > \frac{\varepsilon}{2} \text{ i.o.}(n) \right\} \\ &= 0 \quad (\text{by (2.7)}). \end{aligned}$$

Thus (2.9) holds.

To prove (2.10), it follows from  $b_n \geq b_n^* \uparrow \infty$  and (2.9) that for all  $n \geq 2$ ,

$$\begin{aligned} \frac{\|V_n\|}{b_n} &\leq \frac{\|V_n\|}{b_n^*} \\ &= \frac{\|S_n - S_{n-1}\|}{b_n^*} \\ &\leq \frac{\|S_n\|}{b_n^*} + \frac{\|S_{n-1}\|}{b_n^*} \\ &\leq \frac{\|S_n\|}{b_n^*} + \frac{\|S_{n-1}\|}{b_{n-1}^*} \\ &\rightarrow 0 \text{ a.c.} \end{aligned}$$

Thus for arbitrary  $\varepsilon > 0$

$$P\left\{\frac{\|V_n\|}{b_n} > \varepsilon \text{ i.o.}(n)\right\} = 0$$

whence (2.10) follows from independence and the Borel-Cantelli lemma.  $\square$

The following example shows that Theorem 2.2.1 can fail without the independence assumption.

**Example 2.2.1.** Let  $V$  be a symmetric random element with  $E\|V\|^\alpha = \infty$  for some  $\alpha \in (\frac{1}{2}, 1)$ . Define the sequence of symmetric random elements  $\{V_n, n \geq 1\}$  by

$$V_1 = -V, \quad V_n = \frac{V}{n(n-1)}, \quad n \geq 2.$$

The  $\{V_n, n \geq 1\}$  are of course not independent. Then for  $n \geq 2$ ,

$$S_n = \sum_{i=1}^n V_i$$

$$\begin{aligned}
&= -V + \sum_{i=2}^n \frac{V}{i(i-1)} \\
&= V \left[ -1 + \sum_{i=2}^n \left( \frac{-1}{i} - \frac{-1}{i-1} \right) \right] \\
&= V \left( -1 - \frac{1}{n} + 1 \right) \\
&= -\frac{V}{n}.
\end{aligned}$$

Let  $b_n = \frac{1}{n^{2-\frac{1}{\alpha}}}$ ,  $n \geq 1$ . Then for  $n \geq 2$ ,

$$\frac{S_n}{b_n} = \frac{-V/n}{1/n^{2-\frac{1}{\alpha}}} = \frac{-V}{n^{\frac{1}{\alpha}-1}} \rightarrow 0 \text{ a.c.}$$

since  $0 < \alpha < 1$ . But  $b_n \rightarrow 0$  since  $\alpha > \frac{1}{2}$  and so  $b_n \rightarrow \infty$  fails. Note that

$$b_n^* = \inf_{j \geq n} b_j = 0, \quad n \geq 1$$

and thus the expression  $S_n/b_n^*$  in (2.9) is not defined. Finally, note that for all  $\varepsilon > 0$

$$\begin{aligned}
\sum_{n=2}^{\infty} P\{\|V_n\| > \varepsilon b_n\} &= \sum_{n=2}^{\infty} P\left\{\|V\| > \frac{\varepsilon n(n-1)}{n^{2-\frac{1}{\alpha}}}\right\} \\
&\geq \sum_{n=2}^{\infty} P\left\{\|V\| > \varepsilon n^{1/\alpha}\right\} \\
&= \infty \quad (\text{since } E\|V\|^{\alpha} = \infty)
\end{aligned}$$

whence (2.10) fails.

### 2.3 Sufficient Conditions for a SLLN

#### 2.3.1 SLLNs for Sums of Independent Random Elements in Rademacher Type $p$ Banach Spaces

The following proposition will be utilized in establishing Theorem 2.3.1.

**Proposition 2.3.1.** (Adler, Rosalsky, Taylor (1992a)) *Let  $\{V_n, n \geq 1\}$  be a sequence of independent random elements in a real separable, Rademacher type  $p$  ( $1 \leq p \leq 2$ ) Banach space and let  $\{b_n, n \geq 1\}$  be a sequence of positive constants with  $b_n \uparrow \infty$ . If*

$$\sum_{n=1}^{\infty} \frac{E\|V_n\|^p}{b_n^p} < \infty,$$

*then the SLLN*

$$\frac{1}{b_n} \sum_{i=1}^n (V_i - EV_i) \rightarrow 0 \text{ a.c.}$$

*obtains.*

The main result of this chapter may now be stated and proved. As will be discussed later in more detail, this result is new even when  $\{V_n, n \geq 1\}$  is a sequence of independent random variables.

**Theorem 2.3.1.** *Let  $\{V_n, n \geq 1\}$  be a sequence of independent random elements in a real separable Rademacher type  $p$  ( $1 \leq p \leq 2$ ) Banach space  $\mathcal{X}$ , and let  $\{a_n, n \geq 1\}$  and  $\{b_n, n \geq 1\}$  be sequences of positive constants with  $b_n \uparrow \infty$  such that either*

$$\sum_{n=1}^{\infty} \frac{a_n^p}{b_n^p} < \infty \tag{2.16}$$

or

$$\sum_{i=1}^n a_i = O(b_n) \quad (2.17)$$

hold. Suppose that for some  $\lambda > 0$  and for all  $\varepsilon > 0$

$$\sum_{n=1}^{\infty} P\{||V_n|| > \lambda b_n\} < \infty \quad (2.18)$$

and

$$\sum_{n=1}^{\infty} \frac{1}{b_n^p} E\{||V_n I(\varepsilon a_n < ||V_n|| \leq \lambda b_n) - E(V_n I(\varepsilon a_n < ||V_n|| \leq \lambda b_n))\|^p\} < \infty. \quad (2.19)$$

Then the SLLN

$$\frac{1}{b_n} \sum_{i=1}^n \left( V_i - E(V_i I(||V_i|| \leq \lambda b_i)) \right) \rightarrow 0 \text{ a.c.} \quad (2.20)$$

obtains.

**Remark 2.3.1.** If  $p = 1$ , then (2.16) implies (2.17) by the Kronecker lemma. But in general the conditions (2.16) and (2.17) are independent of each other in the sense that neither implies the other as will now be shown via examples.

- (i) An example of sequences  $\{a_n, n \geq 1\}$  and  $\{b_n, n \geq 1\}$  satisfying (2.16) but not (2.17) is given as follows: Suppose  $1 < p \leq 2$  and  $p^{-1} < \alpha < 1$ . Let  $a_n = 1$  and  $b_n = n^\alpha, n \geq 1$ .
- (ii) An example of sequences  $\{a_n, n \geq 1\}$  and  $\{b_n, n \geq 1\}$  satisfying (2.17) but not (2.16) is given as follows: Suppose  $p = 2$ . Let  $a_n = 2^n$  and  $b_n = \sum_{i=1}^n 2^i = 2^{n+1} - 2, n \geq 1$ .

(iii) Of course if  $1 < p \leq 2$  and  $a_n = 1$  and  $b_n = n$ ,  $n \geq 1$ , then both (2.16) and (2.17) hold.

**Proof of Theorem 2.3.1.** Let  $\varepsilon > 0$  be arbitrary and define

$$V_i^{(1)} = V_i I(|V_i| \leq \varepsilon a_i), \quad i \geq 1,$$

$$V_i^{(2)} = V_i I(\varepsilon a_i < |V_i| \leq \lambda b_i), \quad i \geq 1,$$

and

$$V_i^{(3)} = V_i I(|V_i| > \lambda b_i), \quad i \geq 1.$$

Writing

$$V_i - E(V_i I(|V_i| \leq \lambda b_i)) = (V_i^{(1)} - EV_i^{(1)}) + (V_i^{(2)} - EV_i^{(2)}) + V_i^{(3)}, \quad i \geq 1$$

yields

$$\begin{aligned} & \frac{1}{b_n} \sum_{i=1}^n (V_i - E(V_i I(|V_i| \leq \lambda b_i))) \\ &= \frac{1}{b_n} \sum_{i=1}^n (V_i^{(1)} - EV_i^{(1)}) + \frac{1}{b_n} \sum_{i=1}^n (V_i^{(2)} - EV_i^{(2)}) + \frac{1}{b_n} \sum_{i=1}^n V_i^{(3)}, \quad n \geq 1. \end{aligned} \quad (2.21)$$

Let

$$Z_n = \frac{V_n^{(2)} - EV_n^{(2)}}{b_n}, \quad n \geq 1.$$

Now

$$\begin{aligned}
 & \sum_{n=1}^{\infty} E\|Z_n\|^p \\
 &= \sum_{n=1}^{\infty} \frac{1}{b_n^p} E\|V_n I(\varepsilon a_n < \|V_n\| \leq \lambda b_n) - E(V_n I(\varepsilon a_n < \|V_n\| \leq \lambda b_n))\|^p \\
 &< \infty \quad (\text{by (2.19)}).
 \end{aligned}$$

Thus by Proposition 2.3.1

$$\frac{1}{b_n} \sum_{i=1}^n (V_i^{(2)} - EV_i^{(2)}) \rightarrow 0 \text{ a.e.} \quad (2.22)$$

Next, note that by (2.18)

$$\sum_{n=1}^{\infty} P\{V_n^{(3)} \neq 0\} = \sum_{n=1}^{\infty} P\{\|V_n\| > \lambda b_n\} < \infty$$

and thus  $\{V_n^{(3)}, n \geq 1\}$  is Khintchine equivalent to 0 yielding

$$\frac{1}{b_n} \sum_{i=1}^n V_i^{(3)} \rightarrow 0 \text{ a.e.} \quad (2.23)$$

since  $b_n \uparrow \infty$ .

Now under (2.16),

$$\sum_{n=1}^{\infty} \frac{E\|V_n^{(1)}\|^p}{b_n^p} \leq \sum_{n=1}^{\infty} \frac{\varepsilon^p a_n^p}{b_n^p} < \infty$$

whence by Proposition 2.3.1 we have

$$\frac{1}{b_n} \sum_{i=1}^n \left( V_i^{(1)} - EV_i^{(1)} \right) \rightarrow 0 \text{ a.c.} \quad (2.24)$$

Combining (2.22), (2.23), and (2.24) as in (2.21) establishes (2.20).

Finally, under (2.17), a constant  $0 < C < \infty$  can be chosen so that

$$\sum_{i=1}^n a_i \leq C b_n, \quad n \geq 1. \quad (2.25)$$

Since  $\|V_i^{(1)} - EV_i^{(1)}\| \leq 2\varepsilon a_i$ ,  $i \geq 1$ , we have for all  $n \geq 1$ , with probability one,

$$\begin{aligned} \frac{1}{b_n} \left\| \sum_{i=1}^n \left( V_i^{(1)} - EV_i^{(1)} \right) \right\| &\leq \frac{1}{b_n} \sum_{i=1}^n \|V_i^{(1)} - EV_i^{(1)}\| \\ &\leq \frac{2}{b_n} \sum_{i=1}^n \varepsilon a_i \\ &\leq 2\varepsilon C \quad (\text{by (2.25)}). \end{aligned}$$

Thus

$$\limsup_{n \rightarrow \infty} \frac{1}{b_n} \left\| \sum_{i=1}^n \left( V_i^{(1)} - EV_i^{(1)} \right) \right\| \leq 2\varepsilon C \text{ a.c.} \quad (2.26)$$

Combining (2.22), (2.23), and (2.26) as in (2.21) gives

$$\limsup_{n \rightarrow \infty} \frac{1}{b_n} \left\| \sum_{i=1}^n \left( V_i - E(V_i I(\|V_i\| \leq \lambda b_i)) \right) \right\| \leq 2\varepsilon C \text{ a.c.}$$

thereby establishing (2.20) since  $\varepsilon > 0$  is arbitrary.  $\square$

**Remark 2.3.2.** Observe that (2.16) is weaker for larger  $p$ .

**Remark 2.3.3.** Assertion (2.22) can also be obtained by using Theorem V.7.5 (or Corollary V.7.5) of Woyczyński (1978) and the Kronecker lemma instead of applying Proposition 2.3.1.

**Remark 2.3.4.** It should be noted that the assumption of independence was utilized in the proof of Theorem 2.3.1 only to obtain the convergence of the middle truncation (i.e., to prove that  $\frac{1}{b_n} \sum_{i=1}^n (V_i^{(2)} - EV_i^{(2)}) \rightarrow 0$  a.c.) and to establish (2.24) under (2.16). In Theorem 2.3.4 we will obtain a version of Theorem 2.3.1 (with  $p = 1$ ) without the independence assumption.

**Remark 2.3.5.** It will now be shown that (2.19) holds if

$$\sum_{n=1}^{\infty} \frac{1}{b_n^p} E(\|V_n\|^p I(\varepsilon a_n < \|V_n\| \leq \lambda b_n)) < \infty. \quad (2.27)$$

**Proof.** Note that for all  $n \geq 1$ ,

$$\begin{aligned} & E\|V_n I(\varepsilon a_n < \|V_n\| \leq \lambda b_n) - E(V_n I(\varepsilon a_n < \|V_n\| \leq \lambda b_n))\|^p \\ & \leq 2^p \left( E(\|V_n\|^p I(\varepsilon a_n < \|V_n\| \leq \lambda b_n)) + \|E(V_n I(\varepsilon a_n < \|V_n\| \leq \lambda b_n))\|^p \right) \\ & \leq 2^p \left( E(\|V_n\|^p I(\varepsilon a_n < \|V_n\| \leq \lambda b_n)) + E(\|V_n\|^p I(\varepsilon a_n < \|V_n\| \leq \lambda b_n)) \right) \\ & = 2^{p+1} E(\|V_n\|^p I(\varepsilon a_n < \|V_n\| \leq \lambda b_n)). \end{aligned}$$

Thus

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{b_n^p} E\|V_n I(\varepsilon a_n < \|V_n\| \leq \lambda b_n) - E(V_n I(\varepsilon a_n < \|V_n\| \leq \lambda b_n))\|^p \\ & \leq 2^{p+1} \sum_{n=1}^{\infty} \frac{1}{b_n^p} E(\|V_n\|^p I(\varepsilon a_n < \|V_n\| \leq \lambda b_n)) \end{aligned}$$

$< \infty$ .  $\square$

We now give an example where (2.19) holds but (2.27) fails.

**Example 2.3.1.** Let  $\{X_n, n \geq 1\}$  be a sequence of independent random variables with

$$P\{X_n = n\} = \frac{n^2 - 1}{n^2} = 1 - P\{X_n = 0\}, \quad n \geq 1.$$

Let  $a_n = 1$  and  $b_n = n$ ,  $n \geq 1$ . Setting  $p = 2$  and  $\lambda = 1$  we have

$$\sum_{n=1}^{\infty} \frac{1}{n^2} E(X_n^2 I(\varepsilon < |X_n| \leq n)) = \sum_{n=1}^{\infty} \frac{1}{n^2} \left( n^2 \cdot \frac{n^2 - 1}{n^2} \right) = \sum_{n=1}^{\infty} \frac{n^2 - 1}{n^2} = \infty$$

and thus (2.27) fails. However, (2.19) holds since for  $0 < \varepsilon < 1$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{n^2} E\left(X_n I(\varepsilon < |X_n| \leq n) - E(X_n I(\varepsilon < |X_n| \leq n))\right)^2 \\ &= \sum_{n=1}^{\infty} \frac{1}{n^2} \left[ E(X_n^2 I(\varepsilon < |X_n| \leq n)) - \left(E(X_n I(\varepsilon < |X_n| \leq n))\right)^2 \right] \\ &= \sum_{n=1}^{\infty} \frac{1}{n^2} \left[ \left(n^2 \cdot \frac{n^2 - 1}{n^2}\right) - \left(n \cdot \frac{n^2 - 1}{n^2}\right)^2 \right] \\ &= \sum_{n=1}^{\infty} \frac{n^2 - 1}{n^4} \\ &< \infty. \end{aligned}$$

The next result is a version of Theorem 2.3.1 where the truncated expectations are replaced by true expectations. Of course, in contrast to Theorem 2.3.1, it must

be assumed in Theorem 2.3.2 that the underlying sequence of random elements have expected values.

**Theorem 2.3.2.** *Let  $\{V_n, n \geq 1\}$  be a sequence of independent random elements in a real separable Rademacher type  $p$  ( $1 \leq p \leq 2$ ) Banach space and suppose the  $\{V_n, n \geq 1\}$  all have expected values. Let  $\{a_n, n \geq 1\}$  and  $\{b_n, n \geq 1\}$  be sequences of positive constants with  $b_n \uparrow \infty$  such that either*

$$\sum_{n=1}^{\infty} \frac{a_n^p}{b_n^p} < \infty \quad (2.28)$$

or

$$\sum_{i=1}^n a_i = O(b_n) \quad (2.29)$$

hold. Suppose that for some  $\lambda > 0$  and for all  $\varepsilon > 0$

$$\sum_{n=1}^{\infty} P\{||V_n|| > \lambda b_n\} < \infty, \quad (2.30)$$

$$\frac{1}{b_n} \sum_{i=1}^n E(V_i I(||V_i|| > \lambda b_i)) \rightarrow 0, \quad (2.31)$$

and

$$\sum_{n=1}^{\infty} \frac{1}{b_n^p} E\|V_n I(\varepsilon a_n < ||V_n|| \leq \lambda b_n) - E(V_n I(\varepsilon a_n < ||V_n|| \leq \lambda b_n))\|^p < \infty. \quad (2.32)$$

Then the SLLN

$$\frac{1}{b_n} \sum_{i=1}^n (V_i - EV_i) \rightarrow 0 \text{ a.c.} \quad (2.33)$$

obtains.

**Proof.** By Theorem 2.3.1, the conditions (2.28) (or (2.29)), (2.30), and (2.32) ensure that the SLLN

$$\frac{1}{b_n} \sum_{i=1}^n (V_i - E(V_i I(\|V_i\| \leq \lambda b_i))) \rightarrow 0 \text{ a.c.}$$

obtains. Combining this with the condition (2.31) yields (2.33) thereby proving the result.  $\square$

The following is a generalization of Proposition 2.1.1 to the random element case with general  $1 \leq p \leq 2$ .

**Corollary 2.3.1.** *Let  $\{V_n, n \geq 1\}$  be a sequence of independent random elements in a real separable Rademacher type  $p$  ( $1 \leq p \leq 2$ ) Banach space  $\mathcal{X}$ . If  $\{b_n, n \geq 1\}$  is a sequence of positive constants with  $b_n \uparrow \infty$  and*

$$\sum_{n=1}^{\infty} E \left( \frac{\|V_n\|^p}{\|V_n\|^p + b_n^p} \right) < \infty, \quad (2.34)$$

then the SLLN

$$\frac{1}{b_n} \sum_{i=1}^n (V_i - E(V_i I(\|V_i\| \leq b_i))) \rightarrow 0 \text{ a.c.} \quad (2.35)$$

obtains.

**Proof.** By an argument similar to that of Heyde (1968), the condition (2.34) is equivalent to the pair of conditions

$$\sum_{n=1}^{\infty} P\{||V_n|| > b_n\} < \infty \quad (2.36)$$

and

$$\sum_{n=1}^{\infty} \frac{1}{b_n^p} E(||V_n||^p I(||V_n|| \leq b_n)) < \infty. \quad (2.37)$$

Now (2.37) certainly ensures that

$$\sum_{n=1}^{\infty} \frac{1}{b_n^p} E(||V_n||^p I(\varepsilon a_n < ||V_n|| \leq b_n)) < \infty \quad (2.38)$$

for all  $\varepsilon > 0$  and for any choice of  $\{a_n, n \geq 1\}$ . Choose in particular  $\{a_n, n \geq 1\}$  to satisfy either (2.16) or (2.17). The conclusion (2.35) then follows from (2.36) and (2.38) by Theorem 2.3.1.  $\square$

**Remark 2.3.6.** In Corollary 2.3.1, there is a trade-off between the Rademacher type and the condition (2.34); the larger  $p$ , the stronger is the condition on the underlying Banach space  $\mathcal{X}$  but the weaker is the condition (2.34). To see that (2.34) is weaker for larger  $p$ , since (2.34) is equivalent to the pair of conditions (2.36) and (2.37) as was noted in the proof of Corollary 2.3.1, it suffices to show that (2.37) is weaker for larger  $p$ . Let  $1 \leq p_0 < p \leq 2$  and suppose that  $p_0$  satisfies (2.37). Then

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{b_n^p} E(||V_n||^p I(||V_n|| \leq b_n)) \\ &= \sum_{n=1}^{\infty} \frac{1}{b_n^{p_0}} E \left( ||V_n||^{p_0} \left( \frac{||V_n||}{b_n} \right)^{p-p_0} I(||V_n|| \leq b_n) \right) \end{aligned}$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{b_n^{p_0}} E(|V_n|^{p_0} I(|V_n| \leq b_n)) \\ < \infty$$

and so  $p$  satisfies (2.37).

The following corollary is a version of Corollary 2.3.1 where the truncated expectations are replaced by true expectations. Of course, in contrast to Corollary 2.3.1, it must be assumed that the underlying sequence of random elements have expected values.

**Corollary 2.3.2.** *Let  $\{V_n, n \geq 1\}$  be a sequence of independent random elements in a real separable Rademacher type  $p$  ( $1 \leq p \leq 2$ ) Banach space and suppose the  $\{V_n, n \geq 1\}$  all have expected values. If  $\{b_n, n \geq 1\}$  is a sequence of positive constants with  $b_n \uparrow \infty$ ,*

$$\sum_{n=1}^{\infty} E\left(\frac{|V_n|^p}{|V_n|^p + b_n^p}\right) < \infty,$$

and

$$\frac{1}{b_n} \sum_{i=1}^n E(V_i I(|V_i| > b_i)) \rightarrow 0 \quad (2.39)$$

then the SLLN

$$\frac{1}{b_n} \sum_{i=1}^n (V_i - E(V_i)) \rightarrow 0 \text{ a.c.}$$

obtains.

**Proof.** The result follows immediately from (2.35) and (2.39).  $\square$

The following lemma will be used in the proof of Corollary 2.3.3 and in Example 2.4.6.

**Lemma 2.3.1.** (Adler and Rosalsky (1987)) *Let  $\{Y_n, n \geq 1\}$  be a sequence of random variables which is stochastically dominated by a random variable  $Y$  in the sense that for some constant  $0 < D < \infty$*

$$P\{|Y_n| > t\} \leq DP\{|DY| > t\}, \quad t \geq 0, \quad n \geq 1.$$

*Let  $\{b_n, n \geq 1\}$  be a sequence of positive constants such that*

$$\left( \max_{1 \leq i \leq n} b_i^p \right) \sum_{i=n}^{\infty} \frac{1}{b_i^p} = O(n) \text{ for some } p > 0$$

*and*

$$\sum_{n=1}^{\infty} P\{|Y| > Db_n\} < \infty.$$

*Then for all  $0 < M < \infty$ ,*

$$\sum_{n=1}^{\infty} \frac{1}{b_n^p} E(|Y_n|^p I(|Y_n| \leq Mb_n)) < \infty.$$

The following corollary, due to Adler, Rosalsky, and Taylor (1989), is a random element version of a famous and classical result of Feller (1946). In the special case where  $E\|V_1\|^r < \infty$  for some  $r \in [1, p)$  and  $b_n = n^{1/r}$ ,  $n \geq 1$ , Corollary 2.3.3 reduces to the Marcinkiewicz-Zygmund type SLLN  $(\sum_{i=1}^n V_i)/n^{1/r} \rightarrow 0$  a.c. of Woyczyński (1980). Lai (1974) showed, via example, that in general for a real separable Banach space, the Marcinkiewicz-Zygmund SLLN cannot be extended to the case of i.i.d. mean 0 random elements  $\{V_n, n \geq 1\}$  satisfying  $E\|V_1\|^r < \infty$  for some

$1 < r < 2$ . Lai's example is for the Banach space of real sequences  $b = \{b_j, j \geq 1\}$  converging to 0 with norm  $\|b\| = \max_{j \geq 1} |b_j|$  and this Banach space is only of Rademacher type 1. Thus, the additional condition in Corollary 2.3.3 that  $\mathcal{X}$  is of Rademacher type  $p$  where  $1 < p \leq 2$  is not superfluous.

**Corollary 2.3.3.** (Adler, Rosalsky, and Taylor (1989)) *Let  $\{V_n, n \geq 1\}$  be a sequence of i.i.d. mean 0 random elements in a real separable, Rademacher type  $p$  ( $1 < p \leq 2$ ) Banach space  $\mathcal{X}$  and let  $\{b_n, n \geq 1\}$  be a sequence of positive constants such that*

$$\frac{b_n}{n} \downarrow \text{ and } \frac{b_n}{n^\alpha} \uparrow \text{ for some } \alpha > \frac{1}{p}.$$

If

$$\sum_{n=1}^{\infty} P\{\|V_1\| > b_n\} < \infty, \quad (2.40)$$

then

$$\frac{\sum_{i=1}^n V_i}{b_n} \rightarrow 0 \text{ a.c.} \quad (2.41)$$

**Proof.** Let  $\{a_n, n \geq 1\}$  be a sequence of positive constants satisfying (2.16) or (2.17). Now  $\frac{b_n}{n^\alpha} \uparrow$  and hence  $b_n \uparrow \infty$ . As in the proof of Theorem 4 of Adler, Rosalsky, and Taylor (1989)

$$b_n^p \sum_{i=n}^{\infty} \frac{1}{b_i^p} = O(n).$$

Then by (2.40) and Lemma 2.3.1 with  $Y_n = ||V_n||$ ,  $n \geq 1$  and  $Y = ||V_1||$  we have

$$\sum_{n=1}^{\infty} \frac{1}{b_n^p} E(||V_n||^p I(\varepsilon a_n < ||V_n|| \leq b_n)) \leq \sum_{n=1}^{\infty} \frac{1}{b_n^p} E(||V_n||^p I(||V_n|| \leq b_n)) < \infty.$$

Hence by Remark 2.3.5 and Theorem 2.3.1,

$$\frac{1}{b_n} \sum_{i=1}^n \left( V_i - E(V_i I(||V_i|| \leq b_i)) \right) = \frac{1}{b_n} \sum_{i=1}^n \left( V_i - E(V_i I(||V_i|| \leq b_i)) \right) \rightarrow 0 \text{ a.c.}$$

Thus to achieve the conclusion (2.41), it only remains to show that

$$\frac{1}{b_n} \sum_{i=1}^n E(V_i I(||V_i|| \leq b_i)) \rightarrow 0.$$

To this end, since  $EV_i = 0$ ,  $i \geq 1$ , we have for  $n \geq 1$

$$\begin{aligned} & \frac{1}{b_n} \left\| \sum_{i=1}^n E(V_i I(||V_i|| \leq b_i)) \right\| \\ &= \frac{1}{b_n} \left\| \sum_{i=1}^n E(V_i I(||V_i|| > b_i)) \right\| \\ &\leq \frac{1}{b_n} \sum_{i=1}^n E(||V_i|| I(||V_i|| > b_i)) \\ &\rightarrow 0 \text{ (see Chow and Teicher (1978, pp. 123-124))} \end{aligned}$$

and hence the result is obtained.  $\square$

The next two corollaries are special cases of Theorems 2.3.1 and 2.3.2, respectively with  $a_n = b_n - b_{n-1}$ ,  $n \geq 1$ .

**Corollary 2.3.4.** *Let  $\{V_n, n \geq 1\}$  be a sequence of independent random elements in a real separable Rademacher type  $p$  ( $1 \leq p \leq 2$ ) Banach space  $\mathcal{X}$  and let  $\{b_n, n \geq 1\}$  be a sequence of positive constants with  $b_n \uparrow \infty$  and set  $b_0 = 0$ . Suppose that for some  $\lambda > 0$ , some  $1 \leq p \leq 2$  and all  $\varepsilon > 0$*

$$\sum_{n=1}^{\infty} P\{||V_n|| > \lambda b_n\} < \infty, \quad (2.42)$$

and

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{b_n^p} E\|V_n I(\varepsilon(b_n - b_{n-1}) < ||V_n|| \leq \lambda b_n) \\ - E(V_n I(\varepsilon(b_n - b_{n-1}) < ||V_n|| \leq \lambda b_n))^p\| < \infty. \end{aligned} \quad (2.43)$$

Then the SLLN

$$\frac{1}{b_n} \sum_{i=1}^n \left( V_i - E(V_i I(||V_i|| \leq \lambda b_n)) \right) \rightarrow 0 \text{ a.c.}$$

obtains.

**Proof.** Setting  $a_n = b_n - b_{n-1}$ ,  $n \geq 1$  yields

$$\frac{1}{b_n} \sum_{i=1}^n a_i = \frac{b_n}{b_n} = 1 = O(1). \quad (2.44)$$

The result follows from (2.42), (2.43), and (2.44) by Theorem 2.3.1.  $\square$

**Corollary 2.3.5.** *Let  $\{V_n, n \geq 1\}$  be independent  $\mathcal{L}_1$  random elements in a real separable Rademacher type  $p$  ( $1 \leq p \leq 2$ ) Banach space  $\mathcal{X}$  and let  $\{b_n, n \geq 1\}$  be a sequence of positive constants with  $b_n \uparrow \infty$  and set  $b_0 = 0$ . Suppose that for some*

$\lambda > 0$ , some  $1 \leq p \leq 2$  and all  $\varepsilon > 0$

$$\sum_{n=1}^{\infty} P\{\|V_n\| > \lambda b_n\} < \infty, \quad (2.45)$$

$$\sum_{i=1}^n E(V_i I(\|V_i\| > \lambda b_i)) = o(b_n), \quad (2.46)$$

and

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{b_n^p} E\|V_n I(\varepsilon(b_n - b_{n-1}) < \|V_n\| \leq \lambda b_n) \\ - E(V_n I(\varepsilon(b_n - b_{n-1}) < \|V_n\| \leq \lambda b_n))^p < \infty. \end{aligned} \quad (2.47)$$

Then the SLLN

$$\frac{1}{b_n} \sum_{i=1}^n (V_i - EV_i) \rightarrow 0 \text{ a.c.}$$

obtains.

**Proof.** Setting  $a_n = b_n - b_{n-1}$ ,  $n \geq 1$  yields

$$\frac{1}{b_n} \sum_{i=1}^n a_i = \frac{b_n}{b_n} = 1 = O(1). \quad (2.48)$$

The result follows from (2.45), (2.46), (2.47) and (2.48) by Theorem 2.3.2.  $\square$

### 2.3.2 SLLNs for Compactly Uniformly Integrable Sequences of Independent Random Elements

In the next theorem, a version of Theorem 2.3.2 is obtained by replacing the hypothesis that  $\mathcal{X}$  is of Rademacher type  $p$  by the hypothesis that  $\{V_n, n \geq 1\}$  is

compactly uniformly integrable and by replacing the condition (2.32) by the condition (2.52).

**Theorem 2.3.3.** *Let  $\{V_n, n \geq 1\}$  be a sequence of independent random elements in a real separable Banach space  $\mathcal{X}$ , let  $1 \leq p \leq 2$ , and let  $\{a_n, n \geq 1\}$  and  $\{b_n, n \geq 1\}$  be sequences of positive constants with  $b_n \uparrow$  and  $n = O(b_n)$  such that either*

$$\sum_{n=1}^{\infty} \frac{a_n^p}{b_n^p} < \infty \quad (2.49)$$

or

$$\sum_{i=1}^n a_i = O(b_n) \quad (2.50)$$

hold. Suppose that for some  $\lambda > 0$  and for all  $\varepsilon > 0$

$$\sum_{n=1}^{\infty} P\{||V_n|| > \lambda b_n\} < \infty \quad (2.51)$$

and

$$\sum_{n=1}^{\infty} \frac{1}{b_n^p} E\{||V_n|| I(\varepsilon a_n < ||V_n|| \leq \lambda b_n) - E(||V_n|| I(\varepsilon a_n < ||V_n|| \leq \lambda b_n))\}^p < \infty. \quad (2.52)$$

Then if

$$\{V_n, n \geq 1\} \text{ is compactly uniformly integrable,} \quad (2.53)$$

the SLLN

$$\frac{1}{b_n} \sum_{i=1}^n (V_i - EV_i) \rightarrow 0 \text{ a.c.}$$

obtains.

**Proof.** Let  $C$  be such that  $n \leq Cb_n$ ,  $n \geq 1$ . Clearly  $b_n \uparrow \infty$ . In view of the work of Cuesta and Matrán (1988), Section 4, it suffices to verify that

$$\frac{1}{b_n} \sum_{i=1}^n (||V_i|| - E||V_i||) \rightarrow 0 \text{ a.c.} \quad (2.54)$$

and that

$$\frac{1}{b_n} \sum_{i=1}^n (g(V_i) - Eg(V_i)) \rightarrow 0 \text{ a.c.} \quad (2.55)$$

for every bounded and continuous real function  $g$  on  $\mathcal{X}$ . To prove (2.54), note that since  $\mathbb{R}$  is of Rademacher type  $p$  for every  $p \in [1, 2]$ , we can apply Theorem 2.3.1 to the sequence of random variables  $\{||V_n||, n \geq 1\}$  thereby yielding

$$\frac{1}{b_n} \sum_{i=1}^n \left( ||V_i|| - E(||V_i|| I(||V_i|| \leq \lambda b_i)) \right) \rightarrow 0 \text{ a.c.} \quad (2.56)$$

It will now be shown that

$$E(||V_n|| I(||V_n|| > \lambda b_n)) \rightarrow 0. \quad (2.57)$$

Let  $\varepsilon > 0$  be arbitrary. By (2.53), there exists a compact subset  $K_\varepsilon$  of  $\mathcal{X}$  such that

$$\sup_{n \geq 1} E \left\| V_n I(V_n \notin K_\varepsilon) \right\| \leq \varepsilon.$$

Since  $K_\varepsilon$  is compact, it is bounded (see, e.g., Dugundji (1966) p. 233) and so there exists a constant  $M < \infty$  such that

$$K_\varepsilon \subseteq \{v \in \mathcal{X} : \|v\| \leq M\}.$$

Thus, whenever  $n$  is such that  $\lambda b_n \geq M$ ,

$$[\|V_n\| > \lambda b_n] \subseteq [V_n \notin K_\varepsilon].$$

Then since  $b_n \uparrow \infty$ , for all large  $n$

$$E(\|V_n\| I(\|V_n\| > \lambda b_n)) \leq E(\|V_n\| I(V_n \notin K_\varepsilon)) \leq \varepsilon$$

thereby establishing (2.57) since  $\varepsilon > 0$  is arbitrary. But then

$$\frac{1}{b_n} \sum_{i=1}^n E(\|V_i\| I(\|V_i\| > \lambda b_i)) \leq \frac{C}{n} \sum_{i=1}^n E(\|V_i\| I(\|V_i\| > \lambda b_i)) \rightarrow 0 \quad (2.58)$$

by (2.57) and the Cesàro mean summability theorem. Combining (2.56) and (2.58) yields (2.54).

Next, to verify (2.55), let  $g$  be a bounded and continuous real function defined on  $\mathcal{X}$ . Then, letting  $B = \sup \{|g(v)| : v \in \mathcal{X}\}$ ,

$$\sum_{n=1}^{\infty} \frac{\text{Var}(g(V_n))}{b_n^2} \leq \sum_{n=1}^{\infty} \frac{E(g(V_n))^2}{b_n^2} \leq C^2 B^2 \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

and hence by the Khintchine-Kolmogorov convergence theorem

$$\sum_{n=1}^{\infty} \frac{g(V_n) - E(g(V_n))}{b_n} \text{ converges a.c.}$$

and (2.55) follows immediately by the Kronecker lemma.  $\square$

**Remark 2.3.7.** In the spirit of Remark 2.3.5, the condition (2.52) will hold if

$$\sum_{n=1}^{\infty} \frac{1}{b_n^p} E(\|V_n\|^p I(\varepsilon a_n < \|V_n\| \leq \lambda b_n)) < \infty. \quad (2.59)$$

The following is a version of Corollary 2.3.1 where the hypothesis that  $\mathcal{X}$  is of Rademacher type  $p$  is replaced by the hypothesis that  $\{V_n, n \geq 1\}$  is compactly uniformly integrable. In view of Remark 2.3.6 the result is presented for  $p = 2$  in condition (2.60).

**Corollary 2.3.6.** *Let  $\{V_n, n \geq 1\}$  be a compactly uniformly integrable sequence of independent random elements in a real separable Banach space. If  $\{b_n, n \geq 1\}$  is a sequence of positive constants with  $b_n \uparrow, n = O(b_n)$ , and*

$$\sum_{n=1}^{\infty} E\left(\frac{\|V_n\|^2}{\|V_n\|^2 + b_n^2}\right) < \infty, \quad (2.60)$$

*then the SLLN*

$$\frac{1}{b_n} \sum_{i=1}^n (V_i - EV_i) \rightarrow 0 \text{ a.c.} \quad (2.61)$$

*obtains.*

**Proof.** By the argument of Heyde (1968), the condition (2.60) is equivalent to the pair of conditions

$$\sum_{n=1}^{\infty} P\{\|V_n\| > b_n\} < \infty \quad (2.62)$$

and

$$\sum_{n=1}^{\infty} \frac{1}{b_n^2} E(||V_n||^2 I(||V_n|| \leq b_n)) < \infty. \quad (2.63)$$

Let  $p = 2$  and  $\lambda = 1$ . Now (2.63) certainly ensures that (2.59) holds for all  $\varepsilon > 0$  and for any choice of  $\{a_n, n \geq 1\}$ . Choose in particular  $\{a_n, n \geq 1\}$  to satisfy either (2.49) or (2.50). The conclusion (2.61) then follows from (2.62) and (2.59) by Theorem 2.3.3 and Remark 2.3.7.  $\square$

The last two corollaries follow from Theorem 2.3.3. The first one is due to Adler, Rosalsky, and Taylor (1992a), and the second one to Taylor and Wei (1979).

**Corollary 2.3.7.** (Adler, Rosalsky, Taylor (1992a)) *Let  $\{V_n, n \geq 1\}$  be a sequence of independent random elements in a real separable Banach space and let  $\{b_n, n \geq 1\}$  be a sequence of positive constants with  $b_n \uparrow$  and  $n = O(b_n)$ . If*

$$\{V_n, n \geq 1\} \text{ is compactly uniformly integrable} \quad (2.64)$$

and

$$\sum_{n=1}^{\infty} \frac{E||V_n||^p}{b_n^p} < \infty \text{ for some } 1 \leq p \leq 2, \quad (2.65)$$

then the SLLN

$$\frac{1}{b_n} \sum_{i=1}^n (V_i - EV_i) \rightarrow 0 \text{ a.c.} \quad (2.66)$$

obtains.

**Proof.** Let  $a_n = 1$ ,  $n \geq 1$  and let  $\lambda = 1$ . Note that (2.50) holds by the assumption  $n = O(b_n)$ . Moreover,

$$\begin{aligned} \sum_{n=1}^{\infty} P\{||V_n|| > \lambda b_n\} &= \sum_{n=1}^{\infty} P\{||V_n|| > b_n\} \\ &\leq \sum_{n=1}^{\infty} \frac{E||V_n||^p}{b_n^p} \text{ (by the Markov inequality)} \\ &< \infty \quad \text{(by (2.65)).} \end{aligned}$$

Thus (2.51) holds. Next, note that

$$\sum_{n=1}^{\infty} \frac{1}{b_n^p} E(||V_n||^p I(\varepsilon a_n < ||V_n|| \leq \lambda b_n)) \leq \sum_{n=1}^{\infty} \frac{1}{b_n^p} E||V_n||^p < \infty \quad \text{(by (2.65))}$$

and hence (2.52) holds by Remark 2.3.7. Thus by Theorem 2.3.3 the conclusion (2.66) obtains.  $\square$

The following lemma will be used in the proof of Corollary 2.3.8.

**Lemma 2.3.2.** *If  $\{V_n, n \geq 1\}$  is a sequence of random elements with*

$$\sup_{n \geq 1} E||V_n||^r < \infty \text{ for some } r > 0,$$

*then*

$$\sup_{n \geq 1} E||V_n||^p < \infty \text{ for all } 0 < p < r.$$

**Proof.** Let  $0 < p < r$  and note that for  $n \geq 1$

$$E||V_n||^p = E(||V_n||^p I(||V_n|| \leq 1)) + E(||V_n||^p I(||V_n|| > 1))$$

$$\begin{aligned}
&\leq 1 + E \left( \frac{\|V_n\|^r}{\|V_n\|^{r-p}} I(\|V_n\| > 1) \right) \\
&\leq 1 + E(\|V_n\|^r I(\|V_n\| > 1)) \\
&\leq 1 + E\|V_n\|^r.
\end{aligned}$$

Thus

$$\sup_{n \geq 1} E\|V_n\|^p \leq 1 + \sup_{n \geq 1} E\|V_n\|^r < \infty. \square$$

**Corollary 2.3.8.** (Taylor and Wei (1979)) *Let  $\{V_n, n \geq 1\}$  be a uniformly tight sequence of independent random elements in a real separable Banach space such that*

$$\sup_{n \geq 1} E\|V_n\|^p < \infty \text{ for some } p > 1. \quad (2.67)$$

*Then the SLLN*

$$\frac{1}{n} \sum_{i=1}^n (V_i - EV_i) \rightarrow 0 \text{ a.c.} \quad (2.68)$$

*obtains.*

**Proof.** In view of Lemma 2.3.2 it may be assumed that  $1 < p \leq 2$ . Let  $b_n = n, n \geq 1$ . It follows from (2.67) that the sequence of random variables  $\{\|V_n\|, n \geq 1\}$  is uniformly integrable (see Chow and Teicher (1997, p. 102)). Then this and the uniform tightness hypothesis ensure that the sequence  $\{V_n, n \geq 1\}$  is compactly uniformly integrable (recall the discussion in Chapter 1). Next, by (2.67) there exists a constant  $C < \infty$  such that  $E\|V_n\|^p \leq C, n \geq 1$  implying

$$\sum_{n=1}^{\infty} \frac{E\|V_n\|^p}{n^p} \leq \sum_{n=1}^{\infty} \frac{C}{n^p} < \infty$$

since  $p > 1$ . The conclusion follows from Corollary 2.3.7.  $\square$

### 2.3.3 SLLNs for Banach Space Valued Summands Irrespective of Their Joint Distributions

The last three theorems of this section also establish SLLNs for Banach space valued random elements. In contrast to Theorem 2.3.1, no assumptions are imposed on the underlying Banach space and no assumptions are being made regarding the joint distributions of the random elements  $\{V_n, n \geq 1\}$  whose marginal distributions are constrained solely by (2.70), (2.74), and (2.79), respectively. The only other work that we are aware of pertaining to the SLLN problem where the results obtain irrespective of the joint distributions of the random variables or random elements is that of Sawyer (1966), Chatterji (1970), Martikainen and Petrov (1980), Adler and Rosalsky (1987), and Choi and Sung (1987) (in the random variable case) and Choi and Sung (1987), Adler, Rosalsky, and Taylor (1989), Adler, Rosalsky, and Taylor (1992b), and Adler and Rosalsky (1995) (in the random element case).

The following theorem is a version of Theorem 2.3.1 with  $p = 1$  where the result is obtained without the assumption of independence and where no conditions are imposed on the underlying Banach space. We did not include the condition (2.16) (with  $p = 1$ ) in view of Remark 2.3.1.

**Theorem 2.3.4.** *Let  $\{V_n, n \geq 1\}$  be a sequence of random elements (not necessarily independent) in a real separable Banach space and let  $\{a_n, n \geq 1\}$  and  $\{b_n, n \geq 1\}$  be sequences of positive constants with  $b_n \uparrow \infty$  such that*

$$\sum_{i=1}^n a_i = O(b_n) \quad (2.69)$$

holds. Suppose that for some  $\lambda > 0$  and for all  $\varepsilon > 0$

$$\sum_{n=1}^{\infty} P\{||V_n|| > \lambda b_n\} < \infty \quad (2.70)$$

and

$$\sum_{n=1}^{\infty} \frac{1}{b_n} E\|V_n I(\varepsilon a_n < ||V_n|| \leq \lambda b_n) - E(V_n I(\varepsilon a_n < ||V_n|| \leq \lambda b_n))\| < \infty. \quad (2.71)$$

Then the SLLN

$$\frac{1}{b_n} \sum_{i=1}^n \left( V_i - E(V_i I(||V_i|| \leq \lambda b_i)) \right) \rightarrow 0 \text{ a.c.}$$

obtains irrespective of the joint distributions of the  $\{V_n, n \geq 1\}$ .

**Proof.** Let  $\varepsilon > 0$  be arbitrary and, as in the proof of Theorem 2.3.1, define

$$V_i^{(1)} = V_i I(||V_i|| \leq \varepsilon a_i), \quad i \geq 1$$

$$V_i^{(2)} = V_i I(\varepsilon a_i < ||V_i|| \leq \lambda b_i), \quad i \geq 1$$

and

$$V_i^{(3)} = V_i I(||V_i|| > \lambda b_i), \quad i \geq 1.$$

Now by the Beppo-Levi theorem and (2.71)

$$E \left( \sum_{n=1}^{\infty} \left\| \frac{V_n^{(2)} - EV_n^{(2)}}{b_n} \right\| \right) = \sum_{n=1}^{\infty} E \left\| \frac{V_n^{(2)} - EV_n^{(2)}}{b_n} \right\| < \infty$$

and hence

$$\sum_{n=1}^{\infty} \left\| \frac{V_n^{(2)} - EV_n^{(2)}}{b_n} \right\| < \infty \text{ a.c.}$$

Thus by the Kronecker lemma

$$\frac{1}{b_n} \left\| \sum_{i=1}^n \left( V_i^{(2)} - EV_i^{(2)} \right) \right\| \leq \frac{1}{b_n} \sum_{i=1}^n \| V_i^{(2)} - EV_i^{(2)} \| \rightarrow 0 \text{ a.c.} \quad (2.72)$$

The remainder of the proof follows exactly as in the proof of Theorem 2.3.1 replacing (2.22) with (2.72).  $\square$

The next result is a version of Theorem 2.3.4 where the truncated expectations are replaced by true expectations. Of course, in contrast to Theorem 2.3.4, it must be assumed in Theorem 2.3.5 that the underlying sequence of random elements have expected values.

**Theorem 2.3.5.** *Let  $\{V_n, n \geq 1\}$  be a sequence of random elements (not necessarily independent) in a real separable Banach space and suppose the  $\{V_n, n \geq 1\}$  all have expected values. Let  $\{a_n, n \geq 1\}$  and  $\{b_n, n \geq 1\}$  be sequences of positive constants with  $b_n \uparrow \infty$  such that*

$$\sum_{i=1}^n a_i = O(b_n) \quad (2.73)$$

*holds. Suppose that for some  $\lambda > 0$  and for all  $\varepsilon > 0$*

$$\sum_{n=1}^{\infty} P\{||V_n|| > \lambda b_n\} < \infty, \quad (2.74)$$

$$\frac{1}{b_n} \sum_{i=1}^n E(V_i I(|V_i| > \lambda b_i)) \rightarrow 0, \quad (2.75)$$

and

$$\sum_{n=1}^{\infty} \frac{1}{b_n} E \left| V_n I(\varepsilon a_n < \|V_n\| \leq \lambda b_n) - E(V_n I(\varepsilon a_n < \|V_n\| \leq \lambda b_n)) \right| < \infty. \quad (2.76)$$

Then the SLLN

$$\frac{1}{b_n} \sum_{i=1}^n (V_i - EV_i) \rightarrow 0 \text{ a.c.} \quad (2.77)$$

obtains irrespective of the joint distributions of the  $\{V_n, n \geq 1\}$ .

**Proof.** By Theorem 2.3.4, the conditions (2.73), (2.74), and (2.76) ensure that the SLLN

$$\frac{1}{b_n} \sum_{i=1}^n \left( V_i - E(V_i I(|V_i| \leq \lambda b_i)) \right) \rightarrow 0 \text{ a.c.}$$

obtains. Combining this with the condition (2.75) yields (2.77).  $\square$

The final theorem of this section is also obtained irrespective of the joint distributions of the random elements  $\{V_n, n \geq 1\}$  and without any restrictions on the underlying Banach space. However, as will be shown in Examples 2.4.2 and 2.4.3, Theorems 2.3.6 and 2.3.1 are distinctly different results even when  $\{V_n, n \geq 1\}$  is a sequence of independent random elements and the underlying Banach space is of Rademacher type  $p$  ( $1 \leq p \leq 2$ ). Observe that the assumption (2.78) implies that the norming sequence  $\{b_n, n \geq 1\}$  approaches  $\infty$  very rapidly. An example of a sequence  $\{b_n, n \geq 1\}$  satisfying (2.78) is  $b_n = B^n$ ,  $n \geq 1$  where  $B \in (1, \infty)$ . Another example is described as follows: Let  $\varepsilon > 0$ ,  $b_1 > 0$ , and let  $b_n \geq \varepsilon \sum_{i=1}^{n-1} b_i$ ,  $n \geq 2$ . Then for

$n \geq 2$

$$\sum_{i=1}^n b_i = b_n + \sum_{i=1}^{n-1} b_i \leq b_n + \frac{b_n}{\varepsilon} = \left(1 + \frac{1}{\varepsilon}\right) b_n$$

and so  $\{b_n, n \geq 1\}$  satisfies (2.78).

**Theorem 2.3.6.** *Let  $S_n = \sum_{i=1}^n V_i$ ,  $n \geq 1$  where  $\{V_n, n \geq 1\}$  is a sequence of random elements (not necessarily independent) in a real separable Banach space and let  $\{b_n, n \geq 1\}$  be a sequence of positive constants. If*

$$\sum_{i=1}^n b_i = O(b_n) \quad (2.78)$$

and

$$\sum_{n=1}^{\infty} P\{||V_n|| > \varepsilon b_n\} < \infty \text{ for all } \varepsilon > 0, \quad (2.79)$$

then the SLLN

$$\frac{S_n}{b_n} \rightarrow 0 \text{ a.c.} \quad (2.80)$$

obtains irrespective of the joint distributions of the  $\{V_n, n \geq 1\}$ .

**Proof.** Note at the outset that (2.78) guarantees that  $\sum_{i=1}^{\infty} b_i = \infty$  and hence  $b_n \rightarrow \infty$  again using (2.78). It follows from (2.79) and the Borel-Cantelli lemma that

$$P\left\{\liminf_{n \rightarrow \infty} [||V_n|| \leq \varepsilon b_n]\right\} = 1 \text{ for all } \varepsilon > 0. \quad (2.81)$$

Now (2.78) ensures that there exists a constant  $0 < C < \infty$  such that

$$\sum_{i=1}^n b_i \leq C b_n, \quad n \geq 1. \quad (2.82)$$

Then for arbitrary  $\varepsilon > 0$ , with probability 1,

$$\begin{aligned} \frac{\|S_n\|}{b_n} &\leq \frac{\sum_{i=1}^n \|V_i\|}{b_n} \\ &= \frac{\sum_{i=1}^n \|V_i\| I(\|V_i\| \leq \varepsilon b_i) + \sum_{i=1}^n \|V_i\| I(\|V_i\| > \varepsilon b_i)}{b_n} \\ &\leq \frac{\sum_{i=1}^n \varepsilon b_i}{b_n} + \frac{O(1)}{b_n} \quad (\text{by (2.81)}) \\ &\leq C\varepsilon + o(1) \quad (\text{by (2.82) and } b_n \rightarrow \infty). \end{aligned}$$

Thus

$$\limsup_{n \rightarrow \infty} \frac{\|S_n\|}{b_n} \leq C\varepsilon \text{ a.c.}$$

and since  $\varepsilon > 0$  is arbitrary, the conclusion (2.80) follows.  $\square$

**Remark 2.3.8.** If the sequence  $\{V_n, n \geq 1\}$  is *stochastically dominated* by a random element  $V$  in the sense that for some constant  $0 < D < \infty$

$$P\{\|V_n\| > t\} \leq DP\{\|DV\| > t\}, \quad t \geq 0, \quad n \geq 1,$$

then (2.79) will certainly hold if

$$\sum_{n=1}^{\infty} P\{||V_n|| > \varepsilon b_n\} < \infty \text{ for all } \varepsilon > 0. \quad (2.83)$$

If  $b_n/n^\alpha \uparrow$  for some  $\alpha > 0$ , then (2.83) holds if the series therein converges for *some*  $\varepsilon > 0$ . (See Lemma 3.2.4 of Stout (1974, p. 131) or, for a new simple proof, Rosalsky (1985).)

The following corollary is a version of Corollary 2.3.1 with  $p = 1$  where the result is obtained without the assumption of independence and where no conditions are imposed on the underlying Banach space. Note that in the conclusion (2.85) the centering sequence is the 0 sequence.

**Corollary 2.3.9.** *Let  $\{V_n, n \geq 1\}$  be a sequence of random elements (not necessarily independent) in a real separable Banach space and let  $\{b_n, n \geq 1\}$  be a sequence of positive constants with  $b_n \uparrow \infty$  and*

$$\sum_{n=1}^{\infty} E\left(\frac{||V_n||}{||V_n|| + b_n}\right) < \infty. \quad (2.84)$$

*Then the SLLN*

$$\frac{1}{b_n} \sum_{i=1}^n V_i \rightarrow 0 \text{ a.c.} \quad (2.85)$$

*obtains irrespective of the joint distributions of the  $\{V_n, n \geq 1\}$ .*

**Proof.** By an argument similar to that of Heyde (1968), the condition (2.84) is equivalent to the pair of conditions

$$\sum_{n=1}^{\infty} P\{||V_n|| > b_n\} < \infty \quad (2.86)$$

and

$$\sum_{n=1}^{\infty} \frac{1}{b_n} E(|V_n| I(|V_n| \leq b_n)) < \infty. \quad (2.87)$$

Condition (2.87) certainly ensures that

$$\sum_{n=1}^{\infty} \frac{1}{b_n} E(|V_n| I(\varepsilon a_n < |V_n| \leq b_n)) < \infty \quad (2.88)$$

for all  $\varepsilon > 0$  and for any choice of  $\{a_n, n \geq 1\}$ . Choose in particular  $\{a_n, n \geq 1\}$  to satisfy (2.69). Then

$$\frac{1}{b_n} \sum_{i=1}^n (V_i - E(V_i I(|V_i| \leq b_i))) \rightarrow 0 \text{ a.c.} \quad (2.89)$$

follows (2.86) and (2.88) by Theorem 2.3.4 with  $\lambda = 1$ .

Next,

$$\frac{1}{b_n} \left\| \sum_{i=1}^n E(V_i I(|V_i| \leq b_i)) \right\| \leq \frac{1}{b_n} \sum_{i=1}^n E(|V_i| I(|V_i| \leq b_i)) \rightarrow 0$$

by (2.87) and the Kronecker lemma. Thus

$$\frac{1}{b_n} \sum_{i=1}^n E(V_i I(|V_i| \leq b_i)) \rightarrow 0$$

which when added to (2.89) yields (2.85).  $\square$

## 2.4 Some Interesting Examples/Counterexamples

In this section we present examples illustrating various aspects of the main results in this chapter. The first example revisits Example 2.1.1 and obtains the SLLN

therein by application of Theorem 2.3.1. It will be shown in Examples 2.4.2 and 2.4.3 that Theorems 2.3.1 and 2.3.6 are different results even when the random elements are independent. Example 2.4.4 shows that Theorem 2.3.1 can fail without the hypothesis that the underlying Banach space is of Rademacher type  $p$  and also that Theorem 2.3.3 can fail without the hypotheses that the sequence of random elements is compactly uniformly integrable. The last two examples of this section investigate Theorem 2.3.3 as compared to a similar result of Adler, Rosalsky, and Taylor (1992a).

It will now be shown that the conclusion in Example 2.1.1 which was obtained by first principles can also be obtained utilizing Theorem 2.3.1. Recall that Example 2.1.1 illustrated the limitations of Proposition 2.1.1 and thereby motivated Theorem 2.3.1. In the next chapter it will be shown that Proposition 2.1.1 indeed follows immediately from Theorem 2.3.1.

**Example 2.4.1.** Let  $\{X_n, n \geq 1\}$  be a sequence of independent random variables and  $\{\alpha_n, n \geq 1\}$  be a sequence of constants with  $1 \leq \alpha_n \uparrow \infty$ ,  $\sum_{n=1}^{\infty} \frac{1}{\alpha_n^2} = \infty$ , and

$$P\left\{X_n = \frac{2^n}{\alpha_n}\right\} = P\left\{X_n = \frac{-2^n}{\alpha_n}\right\} = \frac{1}{2}, \quad n \geq 1.$$

To apply Theorem 2.3.1, since the real line is of Rademacher type  $p = 2$ , set  $\mathcal{X} = \mathbb{R}$ ,  $p = 2$ , and  $V_n = X_n$ ,  $n \geq 1$ . Take  $a_n = b_n = 2^n$ ,  $n \geq 1$  and note that condition (2.17) of Theorem 2.3.1 holds. Letting  $\lambda = 1$ , we have since  $\alpha_n \geq 1$  for all  $n \geq 1$  that

$$\sum_{n=1}^{\infty} P\{|V_n| > \lambda b_n\} = \sum_{n=1}^{\infty} P\{|X_n| > 2^n\} = \sum_{n=1}^{\infty} 0 < \infty.$$

Now let  $0 < \varepsilon < 1$  and  $n_0 = \min\{n \geq 3: \alpha_n \geq \varepsilon^{-1}\}$ . Then we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{b_n^p} E \left| V_n I(\varepsilon a_n < \|V_n\| \leq \lambda b_n) - E(V_n I(\varepsilon a_n < \|V_n\| \leq \lambda b_n)) \right|^p \\
&= \sum_{n=1}^{\infty} \frac{1}{2^{2n}} \text{Var}(X_n I(\varepsilon 2^n < |X_n| \leq 2^n)) \\
&= \sum_{n=1}^{n_0-1} \frac{1}{2^{2n}} \text{Var}(X_n I(\varepsilon 2^n < |X_n| \leq 2^n)) + \sum_{n=n_0}^{\infty} \frac{1}{2^{2n}} \text{Var}(X_n I(\varepsilon 2^n < |X_n| \leq 2^n)) \\
&= C + \sum_{n=n_0}^{\infty} 0 \\
&< \infty.
\end{aligned}$$

Hence conditions (2.18) and (2.19) of Theorem 2.3.1 are also satisfied and thus the SLLN

$$\frac{1}{2^n} \sum_{i=1}^n X_i = \frac{1}{2^n} \sum_{i=1}^n (X_i - E(X_i I(|X_i| \leq 2^i))) \rightarrow 0 \text{ a.c.}$$

obtains.

**Remark 2.4.1.** It follows from (2.5) that  $\sum_{n=1}^{\infty} E|X_n|^p/2^{np} = \infty$  and hence the sequence  $\{X_n, n \geq 1\}$  in Example 2.4.1 satisfies the hypotheses of Theorem 2.3.1 but not those of Proposition 2.3.1. This observation is of particular interest in light of the fact that Proposition 2.3.1 played a key role in the proof of Theorem 2.3.1.

Theorems 2.3.1 and 2.3.6 are distinctly different results even when  $\{V_n, n \geq 1\}$  is a sequence of independent random elements and the underlying Banach space is of Rademacher type  $p$  ( $1 \leq p \leq 2$ ) as will be shown in the following two examples. With respect to the real separable Banach space  $\ell_p$  (where  $p \geq 1$ ) of absolute  $p$ -th

power summable real sequences  $v = \{v_j, j \geq 1\}$  with norm  $\|v\| = (\sum_{j=1}^{\infty} |v_j|^p)^{1/p}$ , let  $v^{(n)}$  denote the element having 1 in its  $n$ -th position and 0 elsewhere,  $n \geq 1$ .

In Example 2.4.2, the hypotheses of Theorem 2.3.1 are satisfied but those of Theorem 2.3.6 are not.

**Example 2.4.2.** Let  $1 \leq p \leq 2$  and consider the real separable Rademacher type  $p$  Banach space  $\ell_p$ . Let  $\alpha > 1$  and define a sequence  $\{V_n, n \geq 1\}$  of independent random elements in  $\ell_p$  by requiring the  $\{V_n, n \geq 1\}$  to be independent with

$$P\{V_n = n^\alpha v^{(n)}\} = P\{V_n = -n^\alpha v^{(n)}\} = \frac{1}{2} - \frac{1}{2}P\{V_n = 0\} = \frac{p_n}{2}, \quad n \geq 1$$

where  $\{p_n, n \geq 1\}$  is a sequence of constants with  $p_n \in (0, 1)$ ,  $n \geq 1$  and  $\sum_{n=1}^{\infty} p_n < \infty$ . Let  $b_n = n$ ,  $n \geq 1$  and note that Theorem 2.3.6 cannot be applied since (2.78) is not satisfied with this choice of  $\{b_n, n \geq 1\}$ . To apply Theorem 2.3.1, choose  $a_n = 1$ ,  $n \geq 1$  and  $\lambda = 1$ . Condition (2.17) holds with the choice of sequences  $\{a_n, n \geq 1\}$  and  $\{b_n, n \geq 1\}$  and we have that (2.18) holds since

$$\sum_{n=1}^{\infty} P\{\|V_n\| > \lambda b_n\} = \sum_{n=1}^{\infty} P\{\|V_n\| > n\} = \sum_{n=2}^{\infty} p_n < \infty.$$

Let  $\varepsilon > 0$  and note that (2.19) holds since

$$\sum_{n=2}^{\infty} \frac{1}{n^p} E\|V_n I(\varepsilon < \|V_n\| \leq n) - E(V_n I(\varepsilon < \|V_n\| \leq n))\|^p = \sum_{n=2}^{\infty} 0 < \infty.$$

Hence the hypotheses of Theorem 2.3.1 are satisfied and so the SLLN

$$\frac{1}{n} \sum_{i=1}^n V_i = \frac{1}{n} \sum_{i=1}^n \left( V_i - E(V_i I(\|V_i\| \leq i)) \right) \rightarrow 0 \text{ a.c.}$$

obtains.

In Example 2.4.3, the hypotheses of Theorem 2.3.6 are satisfied but those of Theorem 2.3.1 are not.

**Example 2.4.3.** Let  $1 \leq p \leq 2$  and consider the real separable Rademacher type  $p$  Banach space  $\ell_p$ . Define a sequence  $\{V_n, n \geq 1\}$  of independent random elements in  $\ell_p$  by requiring the  $\{V_n, n \geq 1\}$  to be independent with

$$P\left\{V_n = \frac{B^n}{\log n} v^{(n)}\right\} = P\left\{V_n = -\frac{B^n}{\log n} v^{(n)}\right\} = \frac{1}{2}, \quad n \geq 1$$

where  $B \in (1, \infty)$ . Let  $b_n = B^n$ ,  $n \geq 1$  and note that (2.78) is satisfied. Let  $0 < \varepsilon < 1$  and choose  $n_0$  to be the first integer  $n$  such that  $\frac{1}{\log n} \leq \varepsilon$ . Then

$$\begin{aligned} \sum_{n=1}^{\infty} P\{||V_n|| > \varepsilon B^n\} &= \sum_{n=1}^{n_0-1} P\{||V_n|| > \varepsilon B^n\} + \sum_{n=n_0}^{\infty} P\{||V_n|| > \varepsilon B^n\} \\ &= \sum_{n=1}^{n_0-1} 1 + \sum_{n=n_0}^{\infty} 0 \\ &< \infty. \end{aligned}$$

Hence by Theorem 2.3.6, the SLLN

$$\frac{\sum_{i=1}^n V_i}{B^n} \rightarrow 0 \text{ a.c.}$$

obtains.

Next, let  $a_n = 1$ ,  $n \geq 1$  and note that (2.16) and (2.17) both hold with the choice of sequences  $\{a_n, n \geq 1\}$  and  $\{b_n, n \geq 1\}$  and also (2.18) holds via the same argument presented above. Let  $0 < \varepsilon < B$  and let  $\lambda > 0$ . Now

$$E(V_n I(\varepsilon < ||V_n|| \leq \lambda B^n)) = 0, \quad n \geq 1$$

and hence (2.19) reduces to

$$\sum_{n=1}^{\infty} \frac{1}{B^{np}} E(|V_n|^p I(\varepsilon < |V_n| \leq \lambda B^n)) < \infty.$$

Let  $n_0$  be the first integer  $n$  such that  $\frac{1}{\log n} \leq \lambda$  and note that

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{B^{np}} E(|V_n|^p I(\varepsilon < |V_n| \leq \lambda B^n)) \\ & \geq \sum_{n=n_0}^{\infty} \frac{1}{B^{np}} E(|V_n|^p I(\varepsilon < |V_n| \leq \lambda B^n)) \\ & = \sum_{n=n_0}^{\infty} \frac{1}{B^{np}} \cdot \frac{B^{np}}{(\log n)^p} \\ & = \sum_{n=n_0}^{\infty} \frac{1}{(\log n)^p} \\ & = \infty. \end{aligned}$$

Hence the hypotheses of Theorem 2.3.1 are not satisfied.

The following example of Taylor (1978) and of Adler, Rosalsky, and Taylor (1992a) shows that the Rademacher type  $p$  hypothesis cannot be dispensed with in Theorem 2.3.1. Recall that every real separable Banach space is of Rademacher type (at least) 1 and the  $\ell_p$  spaces are of Rademacher type  $2 \wedge p$ ,  $p \geq 1$ .

**Example 2.4.4.** Consider the real separable Banach space  $\ell_1$ . Define a sequence  $\{V_n, n \geq 1\}$  of independent random elements in  $\ell_1$  by requiring the  $\{V_n, n \geq 1\}$  to be independent with

$$P\{V_n = \sqrt{n}v^{(n)}\} = P\{V_n = -\sqrt{n}v^{(n)}\} = \frac{1}{2} - \frac{1}{2}P\{V_n = 0\} = \frac{1}{2\sqrt{n}}, \quad n \geq 1.$$

Setting  $a_n = 1$ ,  $b_n = n$ ,  $n \geq 1$  and  $\lambda = 1$ , (2.17) holds and

$$\sum_{n=1}^{\infty} P\{|V_n| > \lambda b_n\} = \sum_{n=1}^{\infty} P\{|V_n| > n\} = \sum_{n=1}^{\infty} 0 < \infty$$

establishing (2.18). Also note that for  $1 < p \leq 2$  and  $0 < \varepsilon < 1$  the expression in (2.19) reduces to

$$\sum_{n=1}^{\infty} \frac{1}{n^p} E(|V_n|^p I(\varepsilon < |V_n| \leq n)) = \sum_{n=1}^{\infty} \frac{1}{n^p} \left( \frac{\sqrt{n}^p}{\sqrt{n}} \right) = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{p+1}{2}}} < \infty \quad (2.90)$$

since  $p > 1$  thereby establishing (2.19). It was verified in Taylor (1978, p. 128) that

$$\frac{\|\sum_{i=1}^n V_i\|}{b_n} \xrightarrow{P} 1$$

and hence (2.20) fails. (Alternatively, by the structure of the  $\ell_1$  norm and Kolmogorov's theorem (see Loève (1977, p. 250)) applied to the random variables  $\{\sqrt{n} I(|V_n| = \sqrt{n}), n \geq 1\}$ , it follows that

$$\frac{\|\sum_{i=1}^n V_i\|}{b_n} = \frac{\sum_{i=1}^n \sqrt{i} I(|V_i| = \sqrt{i})}{n} \rightarrow 1 \text{ a.c.}$$

and so (2.20) fails.)

To see that  $\ell_1$  is not of Rademacher type  $p$  for any  $1 < p \leq 2$ , define a sequence  $\{W_n, n \geq 1\}$  of independent random elements in  $\ell_1$  by requiring the  $\{W_n, n \geq 1\}$  to be independent with

$$P\{W_n = v^{(n)}\} = P\{W_n = -v^{(n)}\} = \frac{1}{2}, \quad n \geq 1.$$

Then for  $1 < p \leq 2$ ,

$$E \left\| \sum_{i=1}^n W_i \right\|^p = n^p, \quad \sum_{i=1}^n E \|W_i\|^p = n, \quad n \geq 1$$

whence  $\ell_1$  is not of Rademacher type  $p$  recalling the Hoffmann-Jørgensen and Pisier (1976) characterization of Rademacher type  $p$  Banach spaces. Consequently, Theorem 2.3.1 can fail without the Rademacher type  $p$  hypothesis.

**Remark 2.4.2.** Example 2.4.4 also shows that compact uniform integrability cannot be replaced by uniform tightness in Theorem 2.3.3. To see this, first note that (2.90) ensures (2.52) by Remark 2.3.7. Next, let  $\varepsilon > 0$ , select  $N = N_\varepsilon$  such that  $N^{-1/2} \leq \varepsilon$  and let

$$K_\varepsilon = \{0, v^{(1)}, -v^{(1)}, \sqrt{2}v^{(2)}, -\sqrt{2}v^{(2)}, \dots, \sqrt{N}v^{(N)}, -\sqrt{N}v^{(N)}\}.$$

Then  $K_\varepsilon$  is compact and for  $n \geq 1$

$$P\{V_n \notin K_\varepsilon\} = \begin{cases} 0 < \varepsilon, & \text{if } n \leq N \\ n^{-1/2} < \varepsilon, & \text{if } n > N. \end{cases}$$

Thus,  $\{V_n, n \geq 1\}$  is uniformly tight. However, if  $K$  is any compact set, then  $K$  can contain at most finitely many elements of  $\{\pm \sqrt{n}v^{(n)}, n \geq 1\}$  and so

$$\sup_{n \geq 1} E \|V_n I(V_n \notin K)\| = 1$$

thereby showing that  $\{V_n, n \geq 1\}$  is not compactly uniformly integrable. Therefore (2.53) cannot be replaced by uniform tightness in Theorem 2.3.3.

The last two examples pertain to Theorem 2.3.3 and to Corollary 2.3.7. In Example 2.4.5, the hypotheses of Theorem 2.3.3 hold but not those of Corollary 2.3.7.

**Example 2.4.5.** Let  $\{X_n, n \geq 1\}$  be a sequence of independent random variables with

$$P\{X_n = n^2\} = P\{X_n = -n^2\} = \frac{1}{2n^2 \log n} = \frac{1}{2} - \frac{1}{2}P\{X_n = 0\}, \quad n \geq 1.$$

Note that  $EX_n = 0$ ,  $n \geq 1$ . Since

$$n^2 P\{|X_n| = n^2\} = \frac{1}{\log n} = o(1),$$

it follows from the uniform integrability criterion (see, e.g., Chow and Teicher (1997, p. 94)) that the sequence  $\{X_n, n \geq 1\}$  is (compactly) uniformly integrable. Let  $b_n = n$ ,  $n \geq 1$ . Note that for every  $1 \leq p \leq 2$ ,

$$\sum_{n=1}^{\infty} \frac{E|X_n|^p}{b_n^p} = \sum_{n=1}^{\infty} \frac{n^{2p}}{n^p n^2 \log n} = \sum_{n=1}^{\infty} \frac{1}{n^{2-p} \log n} = \infty$$

since  $2 - p \leq 1$ . Hence Corollary 2.3.7 cannot be applied.

To apply Theorem 2.3.3, let  $a_n = 1$ ,  $n \geq 1$  and  $\lambda = 1$ . Then (2.50) holds. Now

$$\sum_{n=1}^{\infty} P\{|X_n| > \lambda b_n\} = \sum_{n=2}^{\infty} P\{|X_n| > n\} = \sum_{n=2}^{\infty} \frac{1}{n^2 \log n} < \infty$$

and so (2.51) also holds. To verify (2.52), let  $1 \leq p \leq 2$  and note that for all  $\varepsilon \in (0, 1)$ ,

$$\sum_{n=1}^{\infty} \frac{1}{b_n^p} E(|X_n|^p I(\varepsilon a_n < |X_n| \leq \lambda b_n)) = 1 + \sum_{n=2}^{\infty} 0 < \infty.$$

Thus the conditions of Theorem 2.3.3 are satisfied and so

$$\frac{1}{n} \sum_{i=1}^n X_i = \frac{1}{b_n} \sum_{i=1}^n (X_i - EX_i) \rightarrow 0 \text{ a.c.} \quad (2.91)$$

Hence the conclusion (2.66) does indeed hold by Theorem 2.3.3 but not by Corollary 2.3.7. It may be noted that (2.91) can also be obtained by Theorem 2.3.2 recalling that  $\mathbb{R}$  is of Rademacher type  $p$ .

Another example wherein the hypotheses of Theorem 2.3.3 hold but not those of Corollary 2.3.7 is provided by Example 2.4.6 which is the last example of this chapter. It differs from Example 2.4.5 in that:

- Example 2.4.6 concerns a sequence of random elements rather than a sequence of random variables.
- The random elements  $\{V_n, n \geq 1\}$  in Example 2.4.6 are unbounded whereas the random variables  $\{X_n, n \geq 1\}$  in Example 2.4.5 are bounded.
- Theorem 2.3.2 is not necessarily applicable in Example 2.4.6 but, as was noted above, it is applicable in Example 2.4.5.

**Example 2.4.6.** Let  $\{V_n, n \geq 1\}$  be a uniformly tight sequence of independent random elements where  $\|V_n\|$  has distribution given by

$$P\{\|V_n\| \geq x\} = \frac{e}{(\log n)x(\log((\log n)x))^2}, \quad x \geq \frac{e}{\log n}, \quad n \geq 1.$$

Then  $\{V_n, n \geq 1\}$  is stochastically dominated by  $V_1$  and

$$E\|V_1\| = e + \int_e^\infty \frac{e}{x(\log x)^2} dx = e + \left( -e(\log x)^{-1} \Big|_e^\infty \right) = 2e < \infty$$

which combined with the uniform tightness assumption ensure that (see Cuesta and Matrán (1988)) the sequence  $\{V_n, n \geq 1\}$  is compactly uniformly integrable. Let  $a_n = 1$ ,  $b_n = n$ ,  $n \geq 1$  and  $\lambda = 1$ . Now for all  $n \geq 1$ ,

$$\begin{aligned} E||V_n|| &\geq \int_{\frac{e}{\log n}}^{\infty} \frac{e}{(\log n)x(\log((\log n)x))^2} dx \\ &= \int_{\epsilon}^{\infty} \frac{e}{t(\log t)^2(\log n)} dt \\ &= \frac{e}{\log n} \end{aligned}$$

and hence

$$\sum_{n=1}^{\infty} \frac{E||V_n||}{n} = \infty. \quad (2.92)$$

Note that for all  $1 < p \leq 2$  and  $n \geq 1$

$$\begin{aligned} E||V_n||^p &\geq \int_{\frac{e}{\log n}}^{\infty} \frac{x^{p-1}e}{(\log n)x(\log((\log n)x))^2} dx \\ &= \int_{\frac{e}{\log n}}^{\infty} \frac{e}{(\log n)x^{2-p}(\log((\log n)x))^2} dx \\ &= \int_{\epsilon}^{\infty} \frac{e}{(\log n)^p t^{2-p}(\log t)^2} dt \\ &= \infty \quad (\text{since } 2 - p < 1). \end{aligned}$$

Thus, recalling (2.92),

$$\sum_{n=1}^{\infty} \frac{E||V_n||^p}{b_n^p} = \sum_{n=1}^{\infty} \frac{E||V_n||^p}{n^p} = \infty \quad \text{for all } 1 \leq p \leq 2$$

and so Corollary 2.3.7 is not applicable.

Take  $p = 2$ . Then (2.49) and (2.50) both hold. Moreover, since the sequence  $\{V_n, n \geq 1\}$  is stochastically dominated by  $V_1$  (with  $D = 1$ ) and since  $E||V_1|| < \infty$ ,

$$\sum_{n=1}^{\infty} P\{||V_n|| > b_n\} \leq \sum_{n=1}^{\infty} P\{||V_1|| > n\} < \infty \quad (2.93)$$

whence (2.51) holds. Next, it will be shown that (2.52) holds. In view of Remark 2.3.7, it suffices to show that for all  $\varepsilon > 0$

$$\sum_{n=1}^{\infty} \frac{1}{b_n^2} E(||V_n||^2 I(\varepsilon a_n < ||V_n|| \leq \lambda b_n)) \leq \sum_{n=1}^{\infty} \frac{1}{b_n^2} E(||V_n||^2 I(||V_n|| \leq b_n)) < \infty. \quad (2.94)$$

To this end, note that

$$\left( \max_{1 \leq i \leq n} b_i^2 \right) \sum_{i=n}^{\infty} \frac{1}{b_i^2} = n^2 \sum_{i=n}^{\infty} \frac{1}{i^2} = n^2 O\left(\frac{1}{n}\right) = O(n)$$

and recalling (2.93), we have by Lemma 2.3.1 with  $Y_n = ||V_n||$ ,  $n \geq 1$  and  $Y = ||V_1||$  that (2.94) holds. Thus (2.52) is established and so by Theorem 2.3.3

$$\frac{1}{n} \sum_{i=1}^n V_i = \frac{1}{b_n} \sum_{i=1}^n (V_i - EV_i) \rightarrow 0 \text{ a.c.}$$

Hence, as in the previous example, the conclusion (2.66) does indeed hold by Theorem 2.3.3 but not by Corollary 2.3.7.

**Remark 2.4.3.** (i) Note that if the underlying Banach space is not of Rademacher type  $p$  for any  $1 < p \leq 2$  (thus the Banach space is only of Rademacher type 1), then apropos of Example 2.4.6 it will now be shown that condition (2.32) can fail (with  $p = 1$ ) and so Theorem 2.3.2 would not be applicable. Assume that  $V_n$  is

symmetric,  $n \geq 1$ . Now integration by parts yields for all  $n \geq 1$  and  $a > 0$

$$E(||V_n|| I(||V_n|| \leq a)) = \int_0^a P\{||V_n|| > x\} dx - aP\{||V_n|| > a\}.$$

Let  $\varepsilon > 0$  be arbitrary. Then for all large  $n$

$$\begin{aligned} & E(||V_n|| I(\varepsilon a_n < ||V_n|| \leq \lambda b_n)) \\ &= E(||V_n|| I(\varepsilon < ||V_n|| \leq n)) \\ &= E(||V_n|| I(||V_n|| \leq n)) - E(||V_n|| I(||V_n|| \leq \varepsilon)) \\ &= \int_0^n P\{||V_n|| > x\} dx - \int_0^\varepsilon P\{||V_n|| > x\} dx \\ &\quad - nP\{||V_n|| > n\} + \varepsilon P\{||V_n|| > \varepsilon\} \\ &\geq \int_\varepsilon^n P\{||V_n|| > x\} dx - nP\{||V_n|| > n\} \\ &= \int_\varepsilon^n \frac{e}{(\log n)x(\log((\log n)x))^2} dx - nP\{||V_n|| > n\} \\ &= \int_{\varepsilon \log n}^{n \log n} \frac{e}{t(\log t)^2 \log n} dt - nP\{||V_n|| > n\} \\ &= -\frac{e}{\log n} (\log t)^{-1} \Big|_{\varepsilon \log n}^{n \log n} - nP\{||V_n|| > n\} \\ &= \frac{e}{(\log n) \log(\varepsilon \log n)} - \frac{e}{(\log n)(\log n + \log \log n)} - nP\{||V_n|| > n\} \\ &\geq \frac{e}{2(\log n) \log n} - nP\{||V_n|| > n\}. \end{aligned}$$

Thus

$$\begin{aligned}
 & \sum_{n=1}^{\infty} \frac{1}{b_n} E(|V_n| I(\varepsilon a_n < |V_n| \leq \lambda b_n)) \\
 & \geq C + \sum_{n=1}^{\infty} \left( \frac{e}{2n(\log n) \log \log n} - P\{|V_n| > n\} \right) \\
 & = C + \frac{e}{2} \sum_{n=1}^{\infty} \frac{1}{n(\log n) \log \log n} \quad (\text{by (2.93)}) \\
 & = \infty
 \end{aligned}$$

and so (2.32) fails with  $p = 1$ . Since the  $\{V_n, n \geq 1\}$  are symmetric and  $E|V_n| < \infty$ ,  $n \geq 1$ , (2.31) holds. Thus, it is solely the failure of (2.32) which renders Theorem 2.3.2 inapplicable.

(ii) It is interesting to observe that the sequence of random elements  $\{V_n, n \geq 1\}$  in Example 2.4.6 does not satisfy the hypotheses to Theorem 7 of Cuesta and Matrán (1988) solely because

$$E|V_1|^2 = \int_e^{\infty} \frac{e}{(\log x)^2} dx = \infty.$$

**Remark 2.4.4.** In the next chapter, more examples will be presented concerning various aspects of Theorems 2.3.1 and 2.3.6.

CHAPTER 3  
STRONG LAWS OF LARGE NUMBERS FOR SUMS OF RANDOM VARIABLES

### 3.1 Introduction

Attention will now be focused on the case of independent random variables. Since the real line is a Banach space of Rademacher type 2, we can get special cases of the results given in the previous chapter. Examples will be given illustrating many aspects of these results. It will also be shown that Proposition 2.1.1 does indeed follow from Theorem 2.3.1.

### 3.2 Necessary Conditions for a SLLN

The following corollary of Theorem 2.2.1 is a desymmetrized version of Theorem 2.2.1 for the random variable case. Corollary 3.2.1 is originally due to Martikainen (1979) with a proof totally different from the one given below.

**Corollary 3.2.1.** (Martikainen (1979)) *Let  $\{X_n, n \geq 1\}$  be a sequence of independent random variables (not all degenerate) and let  $S_n = \sum_{i=1}^n X_i, n \geq 1$ . If*

$$\frac{S_n - \text{med}(S_n)}{b_n} \rightarrow 0 \text{ a.c.} \quad (3.1)$$

*for some sequence of positive constants  $\{b_n, n \geq 1\}$ , then*

$$b_n \rightarrow \infty \quad (3.2)$$

and

$$\frac{S_n - \text{med}(S_n)}{b_n^*} \rightarrow 0 \text{ a.c.} \quad (3.3)$$

where  $0 < b_n^* \equiv \inf_{j \geq n} b_j \uparrow \infty$ .

**Proof.** Let  $\{S_n^s, n \geq 1\}$  be a symmetrized version of  $\{S_n, n \geq 1\}$ ; that is,  $S_n^s = \sum_{i=1}^n (X_i - \bar{X}_i)$ ,  $n \geq 1$  where  $\{\bar{X}_n, n \geq 1\}$  is a stochastic process independent of  $\{X_n, n \geq 1\}$  with the same distribution as  $\{X_n, n \geq 1\}$ . Then by (3.1)

$$\frac{S_n^s}{b_n} = \frac{S_n - \text{med}(S_n)}{b_n} - \frac{S'_n - \text{med}(S'_n)}{b_n} \rightarrow 0 \text{ a.c.}$$

where  $S'_n = \sum_{i=1}^{\infty} X'_i$ ,  $n \geq 1$  and so by Theorem 2.2.1 we have that (3.2) and

$$\frac{S_n^s}{b_n^*} \rightarrow 0 \text{ a.c.} \quad (3.4)$$

hold. The conclusion (3.3) follows immediately from (3.4) and the strong symmetrization inequality (see, e.g., Loève (1977, p. 259)).  $\square$

### 3.3 Sufficient Conditions for a SLLN

#### 3.3.1 SLLNs for Sums of Independent Random Variables

The main result of this Chapter may now be stated. Theorem 3.3.1 is a special case of Theorem 2.3.1 since the real line is of Rademacher type 2 (and hence is of Rademacher type  $p$  for all  $1 \leq p < 2$ ). It will also be shown that Proposition 2.1.1 follows immediately from Theorem 3.3.1 and therefore from Theorem 2.3.1. A

sufficient condition for (3.8) is of course

$$\sum_{n=1}^{\infty} \frac{1}{b_n^p} E(|X_n|^p I(\varepsilon a_n < |X_n| \leq \lambda b_n)) < \infty.$$

Moreover, if  $\{X_n, n \geq 1\}$  are in  $\mathcal{L}_p$  with a bounded sequence of absolute  $p^{\text{th}}$  moments and if  $\sum_{n=1}^{\infty} b_n^{-p} < \infty$ , then (3.8) is automatic. However, note that there are no moment conditions imposed in Theorem 3.3.1 on the sequence of random variables  $\{X_n, n \geq 1\}$ .

**Theorem 3.3.1.** *Let  $\{X_n, n \geq 1\}$  be a sequence of independent random variables, let  $1 \leq p \leq 2$ , and let  $\{a_n, n \geq 1\}$  and  $\{b_n, n \geq 1\}$  be sequences of positive constants with  $b_n \uparrow \infty$  such that either*

$$\sum_{n=1}^{\infty} \frac{a_n^p}{b_n^p} < \infty \quad (3.5)$$

or

$$\sum_{i=1}^n a_i = O(b_n) \quad (3.6)$$

hold. Suppose that for some  $\lambda > 0$  and all  $\varepsilon > 0$

$$\sum_{n=1}^{\infty} P\{|X_n| > \lambda b_n\} < \infty \quad (3.7)$$

and

$$\sum_{n=1}^{\infty} \frac{1}{b_n^p} E|X_n I(\varepsilon a_n < |X_n| \leq \lambda b_n) - E(X_n I(\varepsilon a_n < |X_n| \leq \lambda b_n))|^p < \infty. \quad (3.8)$$

Then the SLLN

$$\frac{1}{b_n} \sum_{i=1}^n \left( X_i - E(X_i I(|X_i| \leq \lambda b_i)) \right) \rightarrow 0 \text{ a.c.} \quad (3.9)$$

obtains.

**Remark 3.3.1.** Note that when  $p = 2$ , condition (3.8) becomes

$$\sum_{n=1}^{\infty} \frac{1}{b_n^2} \text{Var}(X_n I(\varepsilon a_n < |X_n| \leq \lambda b_n)) < \infty.$$

The next result is a version of Theorem 3.3.1 where the truncated expectations are replaced by true expectations. Of course, in contrast to Theorem 3.3.1, it must be assumed in Theorem 3.3.2 that the underlying sequence of random variables is in  $\mathcal{L}_1$ . Theorem 3.3.2 follows immediately from Theorem 2.3.2 by taking the Banach space to be the real line.

**Theorem 3.3.2.** Let  $\{X_n, n \geq 1\}$  be a sequence of independent  $\mathcal{L}_1$  random variables and let  $1 \leq p \leq 2$ . Let  $\{a_n, n \geq 1\}$  and  $\{b_n, n \geq 1\}$  be sequences of positive constants with  $b_n \uparrow \infty$  such that either

$$\sum_{n=1}^{\infty} \frac{a_n^p}{b_n^p} < \infty \quad (3.10)$$

or

$$\sum_{i=1}^n a_i = O(b_n) \quad (3.11)$$

hold. Suppose that for some  $\lambda > 0$  and for all  $\varepsilon > 0$

$$\sum_{n=1}^{\infty} P\{|X_n| > \lambda b_n\} < \infty, \quad (3.12)$$

$$\sum_{i=1}^n E(X_i I(|X_i| > \lambda b_i)) = o(b_n), \quad (3.13)$$

and

$$\sum_{n=1}^{\infty} \frac{1}{b_n^p} E|X_n I(\varepsilon a_n < |X_n| \leq \lambda b_n) - E(X_n I(\varepsilon a_n < |X_n| \leq \lambda b_n))|^p < \infty. \quad (3.14)$$

Then the SLLN

$$\frac{1}{b_n} \sum_{i=1}^n (X_i - EX_i) \rightarrow 0 \text{ a.c.}$$

obtains.

**Remark 3.3.2.** It will now be shown that Proposition 2.1.1 follows immediately from Theorem 3.3.1. Upon inspection of the hypotheses of Theorem 3.3.1 it is not immediately obvious that Proposition 2.1.1 follows since conditions (3.5) or (3.6) appear in addition to conditions (3.7) and (3.8) which correspond to conditions (2.3) and (2.4) of Proposition 2.1.1.

**Proof of Proposition 2.1.1.** Let  $\lambda = 1$ ,  $p = 2$  and set  $b_0 = 0$  and  $a_n = b_n - b_{n-1}$ ,  $n \geq 1$ . Assume the hypotheses of Proposition 2.1.1. Clearly (2.4) implies (3.8) (with  $p = 2$ ) for every  $\varepsilon > 0$ . Since (3.6) is automatic with the above choice of  $\{a_n, n \geq 1\}$ , Proposition 2.1.1 follows directly from Theorem 3.3.1.  $\square$

**Remark 3.3.3.** Proposition 2.1.1 can of course also be obtained directly from Corollary 2.3.1 with  $\mathcal{X} = \mathbb{R}$  and  $p = 2$ .

The next theorem is a special case of Theorem 2.3.3 where the Banach space  $\mathcal{X}$  is chosen to be the real line.

**Theorem 3.3.3.** *Let  $\{X_n, n \geq 1\}$  be a sequence of independent random variables, let  $1 \leq p \leq 2$ , and let  $\{a_n, n \geq 1\}$  and  $\{b_n, n \geq 1\}$  be sequences of positive constants with  $b_n \uparrow$  and  $n = O(b_n)$  such that either*

$$\sum_{n=1}^{\infty} \frac{a_n^p}{b_n^p} < \infty \quad \text{or} \quad \sum_{i=1}^n a_i = O(b_n)$$

*hold. Suppose that for some  $\lambda > 0$  and for all  $\varepsilon > 0$*

$$\sum_{n=1}^{\infty} P\{|X_n| > \lambda b_n\} < \infty$$

*and*

$$\sum_{n=1}^{\infty} \frac{1}{b_n^p} E[|X_n| I(\varepsilon a_n < |X_n| \leq \lambda b_n) - E(|X_n| I(\varepsilon a_n < |X_n| \leq \lambda b_n))]^p < \infty.$$

*Then if*

$\{X_n, n \geq 1\}$  is uniformly integrable,

*the SLLN*

$$\frac{1}{b_n} \sum_{i=1}^n (X_i - EX_i) \rightarrow 0 \text{ a.c.}$$

*obtains.*

Since the real line is of Rademacher type  $p$  for every  $1 \leq p \leq 2$ , the first corollary of this chapter follows immediately from Corollary 2.3.1 by taking the Banach space to be the real line. Proposition 2.1.1 is the special case  $p = 2$ . However, since the assumption (3.15) is weakest with  $p = 2$  (see Remark 2.3.6), the corollary is strongest when  $p = 2$  (as in Proposition 2.1.1).

**Corollary 3.3.1.** *Let  $\{X_n, n \geq 1\}$  be a sequence of independent random variables and let  $\{b_n, n \geq 1\}$  be a sequence of positive constants with  $b_n \uparrow \infty$  and*

$$\sum_{n=1}^{\infty} E\left(\frac{|X_n|^p}{|X_n|^p + b_n^p}\right) < \infty \text{ for some } 1 \leq p \leq 2. \quad (3.15)$$

*Then the SLLN*

$$\frac{1}{b_n} \sum_{i=1}^n \left( X_i - E(X_i I(|X_i| \leq b_i)) \right) \rightarrow 0 \text{ a.c.}$$

*obtains.*

The next corollary is a special case of Corollary 2.3.2 where the Banach space is chosen to be the real line.

**Corollary 3.3.2.** *Let  $\{X_n, n \geq 1\}$  be a sequence of independent  $\mathcal{L}_1$  random variables and let  $\{b_n, n \geq 1\}$  be a sequence of positive constants with  $b_n \uparrow \infty$ ,*

$$\sum_{n=1}^{\infty} E\left(\frac{|X_n|^p}{|X_n|^p + b_n^p}\right) < \infty \text{ for some } 1 \leq p \leq 2,$$

*and*

$$\sum_{i=1}^n E(X_i I(|X_i| > b_i)) = o(b_n).$$

Then the SLLN

$$\frac{1}{b_n} \sum_{i=1}^n (X_i - EX_i) \rightarrow 0 \text{ a.c.}$$

obtains.

Recall that in the random variable case the Kolmogorov SLLN was generalized by the Marcinkiewicz-Zygmund SLLN which, in turn, was generalized by Feller (1946). As a result of Corollary 2.3.3, Theorem 2.3.1 yields these well-known results. The first corollary presented is the famous result of Feller (1946) which is a special case of Corollary 2.3.3 with  $\mathcal{X} = \mathbb{R}$ ,  $\alpha = 1/(1+\varepsilon)$  (where  $0 \leq \varepsilon < 1$ ), and  $p = 2$ .

**Corollary 3.3.3.** (Feller (1946)) *Let  $S_n = \sum_{i=1}^n X_i$ ,  $n \geq 1$  where  $\{X_n, n \geq 1\}$  is a sequence of i.i.d. mean 0 random variables and let  $\{b_n, n \geq 1\}$  be a sequence of positive constants. Suppose*

$$\frac{b_n}{n} \downarrow \quad \text{and} \quad \frac{b_n}{n^{1/(1+\varepsilon)}} \uparrow \quad \text{for some } 0 \leq \varepsilon < 1.$$

If

$$\sum_{n=1}^{\infty} P\{|X_n| > b_n\} < \infty,$$

then

$$\frac{S_n}{b_n} \rightarrow 0 \text{ a.c.}$$

The next corollary is the famous Marcinkiewicz-Zygmund (1937) SLLN which follows immediately from Corollary 3.3.3.

**Corollary 3.3.4.** (Marcinkiewicz and Zygmund (1937)) *Let  $S_n = \sum_{i=1}^n X_i$ ,  $n \geq 1$  where  $\{X_n, n \geq 1\}$  is a sequence of i.i.d.  $\mathcal{L}_r$  random variables for some  $r \in [1, 2)$ . Then the SLLN*

$$\frac{S_n - nEX_1}{n^{1/r}} \rightarrow 0 \text{ a.c.}$$

*obtains.*

The next lemma is due to Marcinkiewicz and Zygmund (1937). It may also be found in Chow and Teicher (1997, p. 118). This lemma will be utilized to establish the next corollary.

**Lemma 3.3.1.** (Marcinkiewicz and Zygmund (1937)) *For any  $\alpha > r > 0$  and  $\mathcal{L}_r$  random variable  $X$ ,*

$$\sum_{n=1}^{\infty} n^{-\alpha/r} E(|X|^\alpha I(|X| \leq n^{1/r})) < \infty.$$

The following corollary is the renowned Kolmogorov (1933) SLLN for i.i.d.  $\mathcal{L}_1$  random variables. It is of course well known that the Kolmogorov SLLN is a special case of the Marcinkiewicz-Zygmund SLLN (Corollary 3.3.4). Nevertheless, it will be shown in Corollary 3.3.5 that Theorem 3.3.2 does indeed directly yield the Kolmogorov SLLN.

**Corollary 3.3.5.** (Kolmogorov (1933)) *Let  $S_n = \sum_{i=1}^n X_i$ ,  $n \geq 1$  where  $\{X_n, n \geq 1\}$  is a sequence of i.i.d.  $\mathcal{L}_1$  random variables. Then the SLLN*

$$\frac{S_n}{n} \rightarrow EX_1 \text{ a.c.}$$

*obtains.*

**Proof.** We will verify that the hypotheses of Theorem 3.3.2 are satisfied with  $a_n = 1$ ,  $b_n = n$ ,  $n \geq 1$ ,  $p = 2$ , and  $\lambda = 1$ . Firstly, note that (3.10) and (3.11) are immediate and that

$$\sum_{n=1}^{\infty} P\{|X_n| > n\} = \sum_{n=1}^{\infty} P\{|X_1| > n\} < \infty,$$

since  $X_1 \in \mathcal{L}_1$  thereby verifying (3.12).

Next, by Lebesgue dominated convergence theorem, we have

$$E(X_1 I(|X_1| \leq n)) \rightarrow EX_1.$$

Thus

$$E(X_1 I(|X_1| > n)) = EX_1 - E(X_1 I(|X_1| \leq n)) \rightarrow 0,$$

and hence by the Cesàro mean summability theorem

$$\frac{1}{n} \sum_{i=1}^n E(X_1 I(|X_1| > i)) \rightarrow 0.$$

Thus

$$\frac{1}{n} \sum_{i=1}^n E(X_i I(|X_i| > i)) \rightarrow 0$$

thereby verifying (3.13).

Finally, to verify (3.14), note that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{Var}(X_n I(\varepsilon < |X_n| \leq n)) \leq \sum_{n=1}^{\infty} \frac{1}{n^2} E(X_n^2 I(\varepsilon < |X_n| \leq n))$$

$$\begin{aligned}
&\leq \sum_{n=1}^{\infty} \frac{1}{n^2} E(X_n^2 I(|X_n| \leq n)) \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} E(X_1^2 I(|X_1| \leq n)) \\
&< \infty
\end{aligned}$$

by Lemma 3.3.1 with  $\alpha = 2$  and  $r = 1$ . Thus by Theorem 3.3.2 we have

$$\frac{1}{n} \sum_{i=1}^n (X_i - EX_i) \rightarrow 0 \text{ a.c.},$$

and since  $\{X_n, n \geq 1\}$  are i.i.d. the SLLN

$$\frac{S_n}{n} - EX_1 = \frac{S_n - nEX_1}{n} \rightarrow 0 \text{ a.c.}$$

thus obtains.  $\square$

The next two corollaries are special cases of Corollaries 2.3.4 and 2.3.5, respectively with  $\mathcal{X} = \mathbb{R}$ .

**Corollary 3.3.6.** *Let  $\{X_n, n \geq 1\}$  be a sequence of independent random variables and let  $\{b_n, n \geq 1\}$  be a sequence of positive constants with  $b_n \uparrow \infty$  and set  $b_0 = 0$ . Suppose that for some  $\lambda > 0$ , some  $1 \leq p \leq 2$  and all  $\varepsilon > 0$*

$$\sum_{n=1}^{\infty} P\{|X_n| > \lambda b_n\} < \infty,$$

and

$$\sum_{n=1}^{\infty} \frac{1}{b_n^p} E|X_n I(\varepsilon(b_n - b_{n-1}) < |X_n| \leq \lambda b_n)|$$

$$- E(X_n I(\varepsilon(b_n - b_{n-1}) < |X_n| \leq \lambda b_n))^p < \infty.$$

Then the SLLN

$$\frac{1}{b_n} \sum_{i=1}^n (X_i - E(X_i I(|X_i| \leq \lambda b_n))) \rightarrow 0 \text{ a.c.}$$

obtains.

**Corollary 3.3.7.** *Let  $\{X_n, n \geq 1\}$  be independent  $\mathcal{L}_1$  random variables and let  $\{b_n, n \geq 1\}$  be a sequence of positive constants with  $b_n \uparrow \infty$  and set  $b_0 = 0$ . Suppose that for some  $\lambda > 0$ , some  $1 \leq p \leq 2$  and all  $\varepsilon > 0$*

$$\sum_{n=1}^{\infty} P\{|X_n| > \lambda b_n\} < \infty,$$

$$\sum_{i=1}^n E(X_i I(|X_i| > \lambda b_i)) = o(b_n),$$

and

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{b_n^p} E|X_n I(\varepsilon(b_n - b_{n-1}) < |X_n| \leq \lambda b_n) \\ - E(X_n I(\varepsilon(b_n - b_{n-1}) < |X_n| \leq \lambda b_n))^p| < \infty. \end{aligned}$$

Then the SLLN

$$\frac{1}{b_n} \sum_{i=1}^n (X_i - EX_i) \rightarrow 0 \text{ a.c.}$$

obtains.

**Remark 3.3.4.** The proof of Corollary 3.3.6 (resp., Corollary 3.3.7) can also be obtained by direct application of Theorem 3.3.1 (resp., Theorem 3.3.2).

### 3.3.2 SLLNs for Random Variable Summands Irrespective of Their Joint Distributions

The next theorem follows immediately from Theorem 2.3.4 by taking the Banach space to be the real line.

**Theorem 3.3.4.** *Let  $\{X_n, n \geq 1\}$  be a sequence of random variables (not necessarily independent) and let  $\{a_n, n \geq 1\}$  and  $\{b_n, n \geq 1\}$  be sequences of positive constants with  $b_n \uparrow \infty$  such that*

$$\sum_{i=1}^n a_i = O(b_n)$$

*holds. Suppose that for some  $\lambda > 0$  and for all  $\varepsilon > 0$*

$$\sum_{n=1}^{\infty} P\{|X_n| > \lambda b_n\} < \infty$$

*and*

$$\sum_{n=1}^{\infty} \frac{1}{b_n} E|X_n I(\varepsilon a_n < |X_n| \leq \lambda b_n) - E(X_n I(\varepsilon a_n < |X_n| \leq \lambda b_n))| < \infty.$$

*Then the SLLN*

$$\frac{1}{b_n} \sum_{i=1}^n \left( X_i - E(X_i I(|X_i| \leq \lambda b_i)) \right) \rightarrow 0 \text{ a.c.}$$

*obtains irrespective of the joint distributions of the  $\{X_n, n \geq 1\}$ .*

The next result is a version of Theorem 3.3.4 where the truncated expectations are replaced by true expectations. Of course, in contrast to Theorem 3.3.4, it must be assumed in Theorem 3.3.5 that the underlying sequence of random variables is in  $\mathcal{L}_1$ . Theorem 3.3.5 follows immediately from Theorem 2.3.5 by taking the Banach space to be the real line.

**Theorem 3.3.5.** *Let  $\{X_n, n \geq 1\}$  be a sequence of  $\mathcal{L}_1$  random variables (not necessarily independent) and let  $\{a_n, n \geq 1\}$  and  $\{b_n, n \geq 1\}$  be sequences of positive constants with  $b_n \uparrow \infty$  such that*

$$\sum_{i=1}^n a_i = O(b_n)$$

*holds. Suppose that for some  $\lambda > 0$  and for all  $\varepsilon > 0$*

$$\sum_{n=1}^{\infty} P\{|X_n| > \lambda b_n\} < \infty,$$

$$\sum_{i=1}^n E(X_i I(|X_i| > \lambda b_i)) = o(b_n),$$

*and*

$$\sum_{n=1}^{\infty} \frac{1}{b_n} E|X_n I(\varepsilon a_n < |X_n| \leq \lambda b_n) - E(X_n I(\varepsilon a_n < |X_n| \leq \lambda b_n))| < \infty.$$

*Then the SLLN*

$$\frac{1}{b_n} \sum_{i=1}^n (X_i - EX_i) \rightarrow 0 \text{ a.c.}$$

*obtains irrespective of the joint distributions of the  $\{X_n, n \geq 1\}$ .*

The following theorem is a special case of Theorem 2.3.6 where the Banach space is chosen to be the real line.

**Theorem 3.3.6.** *Let  $S_n = \sum_{i=1}^n X_i$ ,  $n \geq 1$  where  $\{X_n, n \geq 1\}$  is a sequence of random variables (not necessarily independent) and let  $\{b_n, n \geq 1\}$  be a sequence of positive constants. If*

$$\sum_{i=1}^n b_i = O(b_n) \quad (3.16)$$

and

$$\sum_{n=1}^{\infty} P\{|X_n| > \varepsilon b_n\} < \infty \text{ for all } \varepsilon > 0, \quad (3.17)$$

then the SLLN

$$\frac{S_n}{b_n} \rightarrow 0 \text{ a.c.} \quad (3.18)$$

obtains irrespective of the joint distributions of the  $\{X_n, n \geq 1\}$ .

The next corollary is a special case of Corollary 2.3.9 where the Banach space is chosen to be the real line.

**Corollary 3.3.8.** *Let  $\{X_n, n \geq 1\}$  be a sequence of random variables (not necessarily independent) and let  $\{b_n, n \geq 1\}$  be a sequence of positive constants with  $b_n \uparrow \infty$  and*

$$\sum_{n=1}^{\infty} E\left(\frac{|X_n|}{|X_n| + b_n}\right) < \infty.$$

Then the SLLN

$$\frac{1}{b_n} \sum_{i=1}^n X_i \rightarrow 0 \text{ a.c.}$$

obtains irrespective of the joint distributions of the  $\{X_n, n \geq 1\}$ .

### 3.4 Some Interesting Examples/Counterexamples

In this section examples will be presented providing insight into the results of this chapter as well as Theorems 2.3.1 and 2.3.6. Examples 3.4.1 and 3.4.2 show that Theorem 3.3.1 is an extension of the motivating result, our Proposition 2.1.1, which was due to Heyde (1968). Examples 3.4.3 and 3.4.4 show that Theorem 3.3.1 (resp., Theorem 2.3.1) can fail if the conditions (3.5) and (3.6) (resp., (2.16) and (2.17)) are dispensed with. A special case is presented in Example 3.4.5 wherein the normed partial sums converge to 1 a.c. The final example (Example 3.4.6) of this section illustrates the sharpness of Theorems 2.3.6 and 3.3.6 as well as demonstrates that Theorem 2.3.6 (resp., Theorem 3.3.6) can fail if the condition (2.79) (resp., (3.17)) is dispensed with.

The next two examples satisfy the hypotheses of Theorem 3.3.1 but not those of Proposition 2.1.1 thereby showing that Theorem 3.3.1 is a bona fide extension of Proposition 2.1.1.

**Example 3.4.1.** Let  $\{p_n, n \geq 1\}$  be a sequence of constants in  $[0, 1]$  with

$$\sum_{n=1}^{\infty} p_n < \infty \tag{3.19}$$

and let  $\{X_n, n \geq 1\}$  be a sequence of independent random variables with distributions given by  $P\{X_1 = 0\} = P\{X_2 = 0\} = 1$  and

$$P\left\{X_n = n^n\right\} = 1 - P\left\{X_n = \frac{2^n}{\log n}\right\} = p_n, \quad n \geq 3.$$

Let  $b_n = 2^n$ ,  $n \geq 1$ , and note that (2.4) is not satisfied in view of

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{b_n^2} E(X_n^2 I(|X_n| \leq b_n)) &= \sum_{n=3}^{\infty} \frac{1}{2^{2n}} E(X_n^2 I(|X_n| \leq 2^n)) \\ &= \sum_{n=3}^{\infty} \frac{1 - p_n}{2^{2n}} \left( \frac{2^{2n}}{(\log n)^2} \right) \\ &= \sum_{n=3}^{\infty} \frac{1 - p_n}{(\log n)^2} \\ &= \sum_{n=3}^{\infty} \frac{1}{(\log n)^2} - \sum_{n=3}^{\infty} \frac{p_n}{(\log n)^2} \\ &= \infty \end{aligned}$$

since the first series diverges and due to (3.19) the second series converges. Hence by Remark 2.1.1, Proposition 2.1.1 cannot be applied. However, setting  $a_n = 2^n$ ,  $n \geq 1$ , condition (3.6) of Theorem 3.3.1 holds. Note that we also have by (3.19)

$$\sum_{n=1}^{\infty} P\{|X_n| > b_n\} = \sum_{n=1}^{\infty} P\{|X_n| > 2^n\} = \sum_{n=3}^{\infty} p_n < \infty$$

and if we let  $0 < \varepsilon < 1$  and let  $n_0 = \min\{n \geq 3: \log n \geq \varepsilon^{-1}\}$  then we have

$$\sum_{n=1}^{\infty} \frac{1}{b_n^2} \text{Var}(X_n I(\varepsilon a_n < |X_n| \leq b_n)) = \sum_{n=1}^{\infty} \frac{1}{2^{2n}} \text{Var}(X_n I(\varepsilon 2^n < |X_n| \leq 2^n))$$

$$\begin{aligned}
&= \sum_{n=1}^{n_0-1} \frac{1}{2^{2n}} \text{Var}(X_n I(\varepsilon 2^n < |X_n| \leq 2^n)) + \sum_{n=n_0}^{\infty} \frac{1}{2^{2n}} \text{Var}(X_n I(\varepsilon 2^n < |X_n| \leq 2^n)) \\
&\leq C + \sum_{n=n_0}^{\infty} \frac{1}{2^{2n}} E(X_n^2 I(\varepsilon 2^n < |X_n| \leq 2^n)) \\
&= C + \sum_{n=n_0}^{\infty} 0 \\
&< \infty.
\end{aligned}$$

Hence conditions (3.7) and (3.8) (with  $\lambda = 1$  and  $p = 2$ ) of Theorem 3.3.1 are also satisfied and thus the SLLN

$$\frac{1}{2^n} \sum_{i=1}^n (X_i - E(X_i I(|X_i| \leq 2^i))) \rightarrow 0 \text{ a.c.}$$

obtains where  $E(X_i I(|X_i| \leq 2^i)) = \frac{2^i(1-p_i)}{\log i}$ ,  $i \geq 3$ .

**Remark 3.4.1.** The sequence of random variables  $\{X_n, n \geq 1\}$  in Example 3.4.1 satisfies the conditions of Theorem 3.3.4 but not those of Corollary 3.3.8. This can be seen by a slight modification of the argument in Example 3.4.1 and the details are left to the reader.

The following example is similar in nature to Example 3.4.1 in that it also reveals the limitations of Proposition 2.1.1. It has the added interest that the random variables  $\{X_n, n \geq 1\}$  are unbounded with  $EX_n = \infty$ ,  $n \geq 1$ .

**Example 3.4.2.** Let  $c > 1$  be such that  $c^n > \log n$  for all  $n \geq 1$  and let  $\{X_n, n \geq 1\}$  be a sequence of independent random variables with corresponding

densities

$$f_n(x) = \begin{cases} \frac{(c^n - \log n) \log n}{c^{2n-1}(c-1)}, & \frac{c^n-1}{\log n} < x \leq \frac{c^n}{\log n} \\ \frac{1}{x^2}, & x > \frac{c^n}{\log n} \\ 0, & \text{elsewhere} \end{cases}, \quad n \geq 1.$$

Note that for all  $n \geq 1$

$$EX_n = \int_0^\infty xf_n(x) dx \geq \int_{\frac{c^n}{\log n}}^\infty \frac{1}{x} dx = \infty.$$

Let  $a_n = b_n = c^n$ ,  $n \geq 1$  and note that

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{b_n^2} E(X_n^2 I(|X_n| \leq b_n)) \\ &= \sum_{n=1}^{\infty} \frac{1}{c^{2n}} E(X_n^2 I(|X_n| \leq c^n)) \\ &= \sum_{n=1}^{\infty} \frac{1}{c^{2n}} \int_0^{c^n} x^2 f_n(x) dx \\ &= \sum_{n=1}^{\infty} \frac{1}{c^{2n}} \left( \int_{\frac{c^n-1}{\log n}}^{\frac{c^n}{\log n}} x^2 \frac{(\log n)(c^n - \log n)}{c^{2n-1}(c-1)} dx + \int_{\frac{c^n}{\log n}}^{c^n} 1 dx \right) \\ &= \sum_{n=1}^{\infty} \frac{1}{c^{2n}} \left( \frac{(\log n)(c^n - \log n)}{c^{2n-1}(c-1)} \frac{x^3}{3} \Big|_{\frac{c^n-1}{\log n}}^{\frac{c^n}{\log n}} + c^n - \frac{c^n}{\log n} \right) \\ &= \sum_{n=1}^{\infty} \frac{1}{c^{2n}} \left( \frac{(c^3 - 1)(c^n - \log n)c^{3n-3}}{3(c-1)(\log n)^2 c^{2n-1}} + \frac{c^n(\log n - 1)}{\log n} \right) \\ &= \sum_{n=1}^{\infty} \left( \frac{c^3 - 1}{3(c-1)} \frac{c^n - \log n}{c^{n+2}(\log n)^2} + \frac{\log n - 1}{c^n \log n} \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{c^3 - 1}{3(c-1)} \left( \sum_{n=1}^{\infty} \frac{1}{c^2(\log n)^2} - \sum_{n=1}^{\infty} \frac{1}{c^{n+2} \log n} \right) + \sum_{n=1}^{\infty} \frac{\log n - 1}{c^n \log n} \\
&= \infty
\end{aligned}$$

since the first series diverges and (since  $c > 1$ ) the second and third series converge. Thus Proposition 2.1.1 does not apply to this example. It will now be shown that Theorem 3.3.1 is applicable. Firstly, condition (3.6) holds due to the definition of  $\{a_n, n \geq 1\}$  and  $\{b_n, n \geq 1\}$ . To verify condition (3.7), note that

$$\sum_{n=1}^{\infty} P\{|X_n| > b_n\} = \sum_{n=1}^{\infty} P\{|X_n| > c^n\} = \sum_{n=1}^{\infty} \int_{c^n}^{\infty} \frac{1}{x^2} dx = \sum_{n=1}^{\infty} \frac{1}{c^n} < \infty$$

since  $c > 1$ . Finally, to verify (3.8) (with  $p = 2$ ), let  $0 < \varepsilon < 1$  and let  $n_0 = \min\{n \geq 3 : \log n \geq \varepsilon^{-1}\}$  and note that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{b_n^2} \text{Var}(X_n I(\varepsilon a_n < |X_n| \leq b_n)) \\
&= \sum_{n=1}^{\infty} \frac{1}{c^{2n}} \text{Var}(X_n I(\varepsilon c^n < |X_n| \leq c^n)) \\
&\leq \sum_{n=1}^{\infty} \frac{1}{c^{2n}} E(X_n^2 I(\varepsilon c^n < |X_n| \leq c^n)) \\
&= \sum_{n=1}^{\infty} \frac{1}{c^{2n}} \int_{\varepsilon c^n}^{c^n} x^2 f_n(x) dx \\
&= \sum_{n=1}^{n_0-1} \frac{1}{c^{2n}} \int_{\varepsilon c^n}^{c^n} x^2 f_n(x) dx + \sum_{n=n_0}^{\infty} \frac{1}{c^{2n}} \int_{\varepsilon c^n}^{c^n} 1 dx \\
&= C + \sum_{n=n_0}^{\infty} \frac{1}{c^{2n}} c^n (1 - \varepsilon)
\end{aligned}$$

$$= C + \sum_{n=n_0}^{\infty} \frac{1-\varepsilon}{c^n}$$

$$< \infty$$

again since  $c > 1$ . Thus by Theorem 3.3.1

$$\frac{1}{c^n} \sum_{i=1}^n \left( X_i - E(X_i I(|X_i| \leq c^i)) \right) \rightarrow 0 \text{ a.c.}$$

These examples again illustrate some limitations of Proposition 2.1.1; on the other hand they satisfy the conditions of Theorem 3.3.1. Note that condition (3.8) (with  $p = 2$ ) of Theorem 3.3.1 is a weaker condition than the corresponding condition (2.4) of Proposition 2.1.1. Although Theorem 3.3.1 contains the additional condition (3.5) or (3.6), we have shown after Theorem 3.3.2 that Proposition 2.1.1 follows immediately from Theorem 3.3.1. Another example of a sequence of random variables satisfying the conditions of Theorem 3.3.1 but not those of Proposition 2.1.1 can be obtained by a slight modification of Example 2.3.1 and the details are left to the reader.

**Remark 3.4.2.** If  $\{X_n, n \geq 1\}$  is a sequence of independent and *symmetric* random variables satisfying the conditions of Theorem 3.3.1 with  $\lambda = 1$  and *with (3.5) holding*, then recalling Remarks 2.3.6 and 2.1.1, it is easy to see that all of the conditions of Proposition 2.1.1 are satisfied. Hence, *in such a case*, Theorem 3.3.1 is not an improvement over Proposition 2.1.1. However, in Examples 2.1.1 and 2.4.1 we presented a sequence of symmetric random variables  $\{X_n, n \geq 1\}$  satisfying the conditions of Theorem 3.3.1 with  $\lambda = 1$  and *with (3.6) holding* but such that the condition (2.1) of Proposition 2.1.1 is not satisfied. (The random variables in the previous two examples were very asymmetric.)

The following two examples show that Theorem 3.3.1 (resp., Theorem 2.3.1) can fail if the conditions (3.5) and (3.6) (resp., (2.16) and (2.17)) are dispensed with.

**Example 3.4.3.** Let  $S_n = \sum_{i=1}^n X_i$ ,  $n \geq 1$  where  $\{X_n, n \geq 1\}$  is a sequence of i.i.d. symmetric bounded random variables with  $EX_1^2 = \sigma^2 > 0$ . Let  $p = 2$ ,  $\lambda = 1$ ,  $a_n = n^{1/4}$  and  $b_n = n^{1/2}$ ,  $n \geq 1$ . Now by the Lévy central limit theorem,

$$n^{-1/2}S_n \xrightarrow{d} N(0, \sigma^2),$$

and thus for arbitrary  $M \geq 1$ ,

$$\begin{aligned} P\left\{\limsup_{n \rightarrow \infty} \frac{S_n}{n^{1/2}} \geq M\right\} &\geq P\left\{\frac{S_n}{n^{1/2}} \geq M \text{ i.o.}(n)\right\} \\ &= P\left\{\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \left[\frac{S_k}{k^{1/2}} \geq M\right]\right\} \\ &= \lim_{n \rightarrow \infty} P\left\{\bigcup_{k=n}^{\infty} \left[\frac{S_k}{k^{1/2}} \geq M\right]\right\} \\ &\geq \limsup_{n \rightarrow \infty} P\left\{\frac{S_n}{n^{1/2}} \geq M\right\} \\ &= \int_M^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-t^2/2\sigma^2} dt \\ &> 0. \end{aligned}$$

Hence by the Kolmogorov 0-1 law,

$$P\left\{\limsup_{n \rightarrow \infty} \frac{S_n}{n^{1/2}} \geq M\right\} = 1.$$

Thus

$$P\left\{\limsup_{n \rightarrow \infty} \frac{S_n}{n^{1/2}} = \infty\right\} = P\left\{\bigcap_{M=1}^{\infty} \left[\limsup_{n \rightarrow \infty} \frac{S_n}{n^{1/2}} \geq M\right]\right\} = 1,$$

and so

$$\limsup_{n \rightarrow \infty} \frac{S_n}{n^{1/2}} = \infty \text{ a.c.} \quad (3.20)$$

Therefore

$$\limsup_{n \rightarrow \infty} \frac{1}{n^{1/2}} \sum_{i=1}^n \left( X_i - E(X_i I(|X_i| \leq \lambda b_i)) \right) = \infty \text{ a.c.}$$

and hence (3.9) fails. In view of  $X_1$  being bounded, (3.7) and (3.8) are automatic since the terms in those series are 0 for all large  $n$ . Finally, note that (3.5) fails since

$$\sum_{n=1}^{\infty} \frac{a_n^2}{b_n^2} = \sum_{n=1}^{\infty} \frac{n^{1/2}}{n} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} = \infty,$$

and (3.6) also fails since

$$\sum_{i=1}^n a_i = \sum_{i=1}^n i^{1/4} = (1 + o(1)) \frac{4}{5} n^{5/4} \neq O(n^{1/2}).$$

**Remark 3.4.3.** Assertion (3.20) also follows immediately from the Hartman and Wintner (1941) law of the iterated logarithm

$$\limsup_{n \rightarrow \infty} \frac{S_n}{\sqrt{n \log \log n}} = \sqrt{2}\sigma \text{ a.c.}$$

In the next example, the sequence  $\{X_n, n \geq 1\}$  is comprised of unbounded random variables.

**Example 3.4.4.** Let  $\{X_n, n \geq 1\}$  be a sequence of i.i.d. standard normal random variables. Let  $p = 2$ ,  $\lambda = 1$ ,  $a_n = n^{1/4}$ , and  $b_n = n^{1/2}$ ,  $n \geq 1$ . Since trivially

$$n^{-1/2} \sum_{i=1}^n X_i \xrightarrow{d} N(0, 1),$$

by the same argument as the previous example, we have

$$\limsup_{n \rightarrow \infty} \frac{1}{n^{1/2}} \sum_{i=1}^n \left( X_i - E(X_i I(|X_i| \leq \lambda b_i)) \right) = \infty \text{ a.c.}$$

and hence (3.9) fails. Now  $EX_1^2 < \infty$  ensures that

$$\sum_{n=1}^{\infty} P\{|X_n| > n^{1/2}\} = \sum_{n=1}^{\infty} P\{|X_1| > n^{1/2}\} < \infty$$

and thus (3.7) holds. Next, choose  $x_0 > 0$  such that

$$x^2 \leq e^{\frac{x^2}{2} - x} \text{ for all } x \geq x_0. \quad (3.21)$$

Let  $0 < \varepsilon < 1$  and let  $n_0$  be the first integer  $n \geq 2$  such that  $\varepsilon n^{1/4} \geq x_0$ . Then

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{n} \text{Var}(X_n I(\varepsilon a_n < |X_n| \leq \lambda b_n)) \\ &= \sum_{n=1}^{\infty} \frac{1}{n} E(X_1^2 I(\varepsilon n^{1/4} < |X_1| \leq n^{1/2})) \\ &= \sum_{n=1}^{\infty} \frac{2}{n \sqrt{2\pi}} \int_{\varepsilon n^{1/4}}^{n^{1/2}} x^2 e^{-x^2/2} dx \\ &= \sum_{n=1}^{n_0-1} \frac{2}{n \sqrt{2\pi}} \int_{\varepsilon n^{1/4}}^{n^{1/2}} x^2 e^{-x^2/2} dx + \sum_{n=n_0}^{\infty} \frac{2}{n \sqrt{2\pi}} \int_{\varepsilon n^{1/4}}^{n^{1/2}} x^2 e^{-x^2/2} dx \end{aligned}$$

$$\begin{aligned}
&\leq C + \sum_{n=n_0}^{\infty} \frac{2}{n\sqrt{2\pi}} \int_{\varepsilon n^{1/4}}^{n^{1/2}} e^{-x} dx \quad (\text{by (3.21)}) \\
&= C + \sum_{n=n_0}^{\infty} \frac{2}{n\sqrt{2\pi}} \left( e^{-\varepsilon n^{1/4}} - e^{-n^{1/2}} \right) \\
&\leq C + \sum_{n=n_0}^{\infty} \frac{2}{n\sqrt{2\pi}} e^{-\varepsilon n^{1/4}} \\
&< \infty
\end{aligned}$$

and thus (3.8) holds. Finally (3.5) and (3.6) fail as was verified in the previous example.

The following example illustrates a special case of Theorem 3.3.2 wherein we define the specific sequences  $\{a_n \equiv EX_n, n \geq 1\}$  and  $\{b_n \equiv \sum_{i=1}^n EX_i, n \geq 1\}$  and the resulting SLLN takes the form  $\frac{S_n}{b_n} \rightarrow 1$  a.c. where  $S_n = \sum_{i=1}^n X_i, n \geq 1$ .

**Example 3.4.5.** Let  $S_n = \sum_{i=1}^n X_i, n \geq 1$  where  $X_1 = 1$  a.c. and  $\{X_n, n \geq 2\}$  is a sequence of independent random variables with corresponding densities

$$f_n(x) = \begin{cases} \frac{(2n^2-1)^2}{8n^3(n-1)}, & 0 < x \leq \frac{4n(n-1)}{2n^2-1} \\ \frac{1}{x^3}, & x > n \\ 0, & \text{elsewhere} \end{cases}, n \geq 2.$$

Note that for  $n \geq 2$

$$\begin{aligned}
EX_n &= \frac{(2n^2-1)^2}{8n^3(n-1)} \int_0^{\frac{4n(n-1)}{2n^2-1}} x dx + \int_n^{\infty} \frac{1}{x^2} dx \\
&= \frac{(2n^2-1)^2}{8n^3(n-1)} \cdot \frac{16n^2(n-1)^2}{2(2n^2-1)^2} + \frac{1}{n} \\
&= \frac{n-1}{n} + \frac{1}{n}
\end{aligned}$$

$$= 1.$$

Setting  $a_n = EX_n = 1$ ,  $b_n = \sum_{i=1}^n EX_i = n$ ,  $n \geq 1$  and  $\lambda = 1$  we have that

$$\sum_{n=1}^{\infty} P\{|X_n| > b_n\} = \sum_{n=2}^{\infty} P\{|X_n| > n\} = \sum_{n=2}^{\infty} \int_n^{\infty} \frac{1}{x^3} dx = \sum_{n=2}^{\infty} \frac{1}{2n^2} < \infty.$$

Also note that

$$\sum_{n=1}^{\infty} \frac{1}{b_n} E(X_n I(X_n > n)) = \sum_{n=2}^{\infty} \frac{1}{n} \int_n^{\infty} \frac{1}{x^2} dx = \sum_{n=2}^{\infty} \frac{1}{n^2} < \infty$$

and hence by the Kronecker lemma

$$\frac{1}{b_n} \sum_{i=1}^n E(X_i I(X_i > i)) \rightarrow 0.$$

Finally we have for all  $\varepsilon > 0$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{b_n^2} \text{Var}(X_n I(\varepsilon a_n < |X_n| \leq b_n)) &\leq \sum_{n=1}^{\infty} \frac{1}{n^2} E(X_n^2 I(X_n \leq n)) \\ &= 1 + \sum_{n=2}^{\infty} \frac{1}{n^2} \frac{(2n^2 - 1)^2}{8n^3(n-1)} \int_0^{\frac{4n(n-1)}{2n^2-1}} x^2 dx \\ &= 1 + \sum_{n=2}^{\infty} \frac{1}{n^2} \cdot \frac{8(n-1)^2}{3(2n^2-1)} \\ &< \infty \end{aligned}$$

and thus by Theorem 3.3.2 the SLLN

$$\frac{1}{b_n} \sum_{i=1}^n (X_i - EX_i) \rightarrow 0 \text{ a.c.}$$

obtains. Hence we have

$$\frac{S_n}{b_n} \rightarrow 1 \text{ a.c.}$$

The following result of Rosalsky (1993) will be used in the ensuing example.

**Proposition 3.4.1.** (Rosalsky (1993)) *Let  $S_n = \sum_{i=1}^n X_i$ ,  $n \geq 1$  where  $\{X_n, n \geq 1\}$  is a sequence of independent random variables and let  $\{B_n, n \geq 1\}$  be a sequence of positive constants with  $B_n \rightarrow \infty$ .*

(i) *Suppose there exist constants  $0 < M < \infty$  and  $\delta > 0$  such that for all large  $n$*

$$P\left\{\frac{S_{n-1}}{B_n} \geq -M\right\} \geq \delta.$$

*If*

$$\limsup_{n \rightarrow \infty} \frac{X_n}{B_n} = \infty \text{ a.c.,}$$

*then*

$$\limsup_{n \rightarrow \infty} \frac{S_n}{B_n} = \infty \text{ a.c.}$$

(ii) *Suppose there exist constants  $0 < M < \infty$  and  $\delta > 0$  such that for all large  $n$*

$$P\left\{\frac{S_{n-1}}{B_n} \leq M\right\} \geq \delta.$$

*If*

$$\liminf_{n \rightarrow \infty} \frac{X_n}{B_n} = -\infty \text{ a.c.,}$$

then

$$\liminf_{n \rightarrow \infty} \frac{S_n}{B_n} = -\infty \text{ a.c.}$$

The following example concerns the SLLN problem for geometrically weighted i.i.d. random variables. (Iterated logarithm type results for geometrically weighted i.i.d. random variables were obtained by Rosalsky (1981).) The conclusions (3.22), (3.24) (with  $A_n = n^{1/q}$ ,  $n \geq 1$ ), and (3.25) (or (3.26)) demonstrate the sharpness of Theorems 2.3.6 and 3.3.6. Moreover, this example demonstrates that Theorem 2.3.6 (resp., Theorem 3.3.6) can fail if the condition (2.79) (resp., (3.17)) is dispensed with. Indeed, in the case of *independent* summands  $\{V_n, n \geq 1\}$  and a norming sequence satisfying (2.78), the condition (2.79) is necessary for (2.80) as will now be shown: If the series of (2.79) diverges for some  $\varepsilon > 0$ , then by the Borel-Cantelli lemma

$$P\left\{\limsup_{n \rightarrow \infty} \frac{\|V_n\|}{b_n} \geq \varepsilon\right\} \geq P\{\|V_n\| > \varepsilon b_n \text{ i.o.}(n)\} = 1$$

and noting that  $b_{n-1} = O(b_n)$  by (2.78) we have

$$\begin{aligned} \varepsilon &\leq \limsup_{n \rightarrow \infty} \frac{\|V_n\|}{b_n} \\ &= \limsup_{n \rightarrow \infty} \frac{\|S_n - S_{n-1}\|}{b_n} \\ &\leq \limsup_{n \rightarrow \infty} \left( \frac{\|S_n\|}{b_n} + \frac{\|S_{n-1}\|}{b_n} \right) \\ &\leq \limsup_{n \rightarrow \infty} \left( \frac{\|S_n\|}{b_n} + C \frac{\|S_{n-1}\|}{b_{n-1}} \right) \\ &\leq \limsup_{n \rightarrow \infty} \frac{\|S_n\|}{b_n} + C \limsup_{n \rightarrow \infty} \frac{\|S_{n-1}\|}{b_{n-1}} \end{aligned}$$

$$= C \limsup_{n \rightarrow \infty} \frac{\|S_n\|}{b_n} \text{ a.c.}$$

Thus (2.80) fails.

**Example 3.4.6.** Let  $\{Y_n, n \geq 1\}$  be a sequence of i.i.d. random variables with  $E|Y_1|^p < \infty$  for some  $p > 0$ . Set  $X_n = a^n Y_n, n \geq 1$  where  $a > 1$ ,  $S_n = \sum_{i=1}^n X_i, n \geq 1$ , and  $b_n = a^n n^{1/p}, n \geq 1$ . Then (3.16) holds. Now for arbitrary  $\varepsilon > 0$ ,  $E\left|\frac{Y_1}{\varepsilon}\right|^p < \infty$  ensures that

$$\sum_{n=1}^{\infty} P\{|X_n| > \varepsilon b_n\} = \sum_{n=1}^{\infty} P\left\{\left|\frac{Y_1}{\varepsilon}\right| > n^{1/p}\right\} < \infty$$

whence by Theorem 3.3.6 the SLLN

$$\frac{S_n}{b_n} = \frac{\sum_{i=1}^n a^i Y_i}{a^n n^{1/p}} \rightarrow 0 \text{ a.c.} \quad (3.22)$$

obtains.

Next, suppose that  $E|Y_1|^q = \infty$  for some  $q > 0$  (necessarily  $q > p$ ). Then  $b'_n \equiv a^n n^{1/q}, n \geq 1$  satisfies (3.16). Now for arbitrary  $M > 0$ ,  $E\left|\frac{Y_1}{M}\right|^q = \infty$  ensures that

$$\sum_{n=1}^{\infty} P\{|X_n| > M a^n n^{1/q}\} = \sum_{n=1}^{\infty} P\left\{\left|\frac{Y_1}{M}\right| > n^{1/q}\right\} = \infty$$

whence  $\{b'_n, n \geq 1\}$  does not satisfy (3.17) and by the Borel-Cantelli lemma

$$P\left\{\limsup_{n \rightarrow \infty} \frac{|X_n|}{a^n n^{1/q}} \geq M\right\} \geq P\{|X_n| > M a^n n^{1/q} \text{ i.o.}(n)\} = 1.$$

Thus,

$$\begin{aligned}
M &\leq \limsup_{n \rightarrow \infty} \frac{|X_n|}{a^n n^{1/q}} \\
&= \limsup_{n \rightarrow \infty} \frac{|S_n - S_{n-1}|}{a^n n^{1/q}} \\
&\leq \limsup_{n \rightarrow \infty} \left( \frac{|S_n|}{a^n n^{1/q}} + \frac{|S_{n-1}|}{a^n n^{1/q}} \right) \\
&\leq \limsup_{n \rightarrow \infty} \frac{|S_n|}{a^n n^{1/q}} + \limsup_{n \rightarrow \infty} \frac{|S_{n-1}|}{a^{n-1} (n-1)^{1/q}} \\
&= 2 \limsup_{n \rightarrow \infty} \frac{|S_n|}{a^n n^{1/q}} \text{ a.c.}
\end{aligned}$$

implying

$$\limsup_{n \rightarrow \infty} \frac{|S_n|}{a^n n^{1/q}} \geq \frac{M}{2} \text{ a.c.}$$

Since  $M > 0$  is arbitrary,

$$\limsup_{n \rightarrow \infty} \frac{|\sum_{i=1}^n a^i Y_i|}{a^n n^{1/q}} = \limsup_{n \rightarrow \infty} \frac{|S_n|}{a^n n^{1/q}} = \infty \text{ a.c.} \quad (3.23)$$

and so (3.18) fails in an extreme way.

It will now be shown that a conclusion more precise than (3.23) can be obtained by applying Proposition 3.4.1. Note that for any numerical sequence  $0 < A_n \rightarrow \infty$  (no matter how slowly), for arbitrary  $\varepsilon > 0$ , setting  $r = p \wedge 1$

$$\begin{aligned}
P \left\{ \frac{|S_n|}{a^n A_n} > \varepsilon \right\} &\leq P \left\{ \frac{\sum_{i=1}^n a^{ri} |Y_i|^r}{a^n A_n^r} > \varepsilon^r \right\} \\
&\leq \frac{\sum_{i=1}^n a^{ri} E|Y_1|^r}{\varepsilon^r a^n A_n^r} \quad (\text{by the Markov inequality})
\end{aligned}$$

$$= O\left(\frac{1}{A_n^r}\right)$$

$$= o(1).$$

Thus the WLLN

$$\frac{\sum_{i=1}^n a^i Y_i}{a^n A_n} = \frac{S_n}{a^n A_n} \xrightarrow{P} 0 \quad (3.24)$$

holds. Then setting  $A_n = a(n+1)^{1/q}$ ,  $n \geq 1$ , we have for all  $n \geq 2$

$$P\left\{\frac{S_{n-1}}{a^n n^{1/q}} \geq -1\right\} \geq P\left\{\frac{|S_{n-1}|}{a^{n-1} (an^{1/q})} \leq 1\right\} \rightarrow 1.$$

Since  $E|Y_1|^q = \infty$ , either  $E(Y_1^+)^q = \infty$  or  $E(Y_1^-)^q = \infty$ . If  $E(Y_1^+)^q = \infty$ , then by a standard application of the Borel-Cantelli lemma

$$\limsup_{n \rightarrow \infty} \frac{X_n}{a^n n^{1/q}} = \limsup_{n \rightarrow \infty} \frac{Y_n}{n^{1/q}} = \infty \text{ a.c.}$$

whence by Proposition 3.4.1(i)

$$\limsup_{n \rightarrow \infty} \frac{\sum_{i=1}^n a^i Y_i}{a^n n^{1/q}} = \limsup_{n \rightarrow \infty} \frac{S_n}{a^n n^{1/q}} = \infty \text{ a.c.} \quad (3.25)$$

Similarly, if  $E(Y_1^-)^q = \infty$ , then

$$\liminf_{n \rightarrow \infty} \frac{\sum_{i=1}^n a^i Y_i}{a^n n^{1/q}} = \liminf_{n \rightarrow \infty} \frac{S_n}{a^n n^{1/q}} = -\infty \text{ a.c.} \quad (3.26)$$

Of course, (3.25) and (3.26) are each more precise than (3.23).

**Remark 3.4.4.** A perusal of the argument in Example 3.4.6 reveals that the independence hypothesis was not used in obtaining (3.22) and (3.24).

## CHAPTER 4 SUMMARY AND IDEAS FOR FUTURE RESEARCH

### 4.1 Summary

In this study we have presented numerous results pertaining to the SLLN problem for sums of Banach space valued random elements (as well as the special case of sums of random variables). In Theorem 2.2.1 we obtained necessary conditions for a SLLN for sums of independent random elements. Conclusions (2.9) and (2.10) are extensions of a result due to Martikainen (1979) in the random variable case but contain the added assumption that the random elements are symmetric. The exact result of Martikainen (1979) is presented as Corollary 3.2.1 and is obtained via Theorem 2.2.1.

Theorem 2.3.1 provides an extension of Proposition 2.1.1 wherein the hypotheses of Proposition 2.1.1 are weakened and pertain to the more general case of Banach space valued random elements. Theorems 2.3.1 and 2.3.2 both impose the condition that the Banach space is of Rademacher type  $p$ ,  $1 \leq p \leq 2$  (which is automatic in the special case when the Banach space is the real line). Theorem 2.3.3 is similar in nature to Theorem 2.3.2. However, in Theorem 2.3.3 we substitute the condition that the sequence of random elements is compactly uniformly integrable for the condition that the Banach space is of Rademacher type  $p$ ,  $1 \leq p \leq 2$ . Theorems 2.3.4, 2.3.5, and 2.3.6 obtain SLLNs where there are no assumptions made regarding the joint distributions of the random elements. Each of these theorems provides a new result even in the special case of random variables; that is, in the special case where the underlying Banach space is the real line. All of these special cases are presented in Section 3.3.

As mentioned in Chapter 3, many well-known results are corollaries or special cases of the results in the current work. Among them are:

- Proposition 2.1.1 which is due to Heyde (1968)
- Corollary 3.3.3 which is a famous result of Feller (1946)
- Corollary 3.3.4 which is the Marcinkiewicz-Zygmund (1937) SLLN
- Corollary 3.3.5 which is the celebrated Kolmogorov (1933) SLLN
- Corollaries 2.3.3 and 2.3.7 which are due to Adler, Rosalsky, and Taylor (1989) and (1992a), respectively
- Corollary 2.3.8 which is due to Taylor and Wei (1979).

In Sections 2.4 and 3.4 we have presented numerous examples concerning various aspects of the results contained in the current work. Examples were given illustrating the sharpness of the results as well as demonstrating the distinctions among them. For some of the aforementioned corollaries, we presented examples illustrating that these results are indeed extended by the current work.

## 4.2 Ideas For Future Research

Some ideas for future research will now be discussed.

1. Since almost certain convergence implies convergence in probability, all of the SLLN results in this study are automatically WLLN results as well. It is thus natural to inquire as to whether the hypotheses of a SLLN result can be weakened so as to obtain the corresponding WLLN. We would hope that any such result would be sharp in the sense that if the hypotheses are further weakened, the result can fail.

2. A real separable Banach space  $\mathcal{X}$  is said to be of *martingale type p* ( $1 \leq p \leq 2$ ) if for all martingale difference sequences  $\{V_n, n \geq 1\}$  in  $\mathcal{X}$

$$E \left\| \sum_{i=1}^n V_i \right\|^p \leq \sum_{i=1}^n E \|V_i\|^p \quad (4.1)$$

for all  $n \geq 1$ . Note the formal similarity between (4.1) and (1.2) (the Hoffmann-Jørgensen and Pisier (1976) characterization of Rademacher type  $p$  Banach spaces). If  $\mathcal{X}$  is of martingale type  $p$ , then  $\mathcal{X}$  is of Rademacher type  $p$ . (See Scalora (1961) for a complete development of conditional expectation of random elements and of Banach space valued martingales including martingale convergence theorems.) It is reasonable to conjecture that Theorem 2.3.1 can be extended to hold for a martingale difference sequence in a martingale type  $p$  Banach space. In the case of a sequence of independent random elements, truncation preserves independence. However, in the case of a martingale difference sequence, the martingale property is in general not preserved by truncation which may thus introduce serious complications.

3. Investigate whether Theorem 2.3.2 (resp., Theorem 2.3.3) holds with (2.32) replaced by (2.52) (resp., (2.52) replaced by (2.32)). When comparing these two conditions, they appear to be very similar but there is, however, a major distinction. Note that in condition (2.32) of Theorem 2.3.2 the inner portion,

$$V_n I(\varepsilon a_n < \|V_n\| \leq \lambda b_n) - E(V_n I(\varepsilon a_n < \|V_n\| \leq \lambda b_n)),$$

is a random element. In contrast, in condition (2.52) of Theorem 2.3.3 the inner portion,

$$||V_n||I(\varepsilon a_n < ||V_n|| \leq \lambda b_n) - E(||V_n||I(\varepsilon a_n < ||V_n|| \leq \lambda b_n)),$$

is a random variable. This distinction has proven to be quite difficult to overcome in the present context. It is conceivable that such a substitution can be made in each theorem. However, that may require additional conditions on the sequence of random elements or on the underlying Banach space. But it should be noted that since for any random element  $V$  with  $E||V|| < \infty$  we have

$$E||V - E||V|||^2 \leq E||V - EV||^2 \leq E||V - EV||^2,$$

the implication (2.32)  $\Rightarrow$  (2.52) holds (at least) for  $p = 2$ .

4. In view of Theorem 2.3.4, it seems reasonable to conjecture when  $p = 1$  that in Theorem 2.3.3 the assumption of independence as well as condition (2.53) may not be needed.

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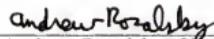
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W. A. Woyczyński, On Marcinkiewicz-Zygmund laws of large numbers in Banach spaces and related rates of convergence, *Probab. Math. Statist.* **1** (1980), 117–131.

## BIOGRAPHICAL SKETCH

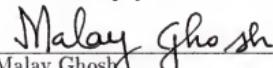
Amy M. Cantrell was born in Easley, South Carolina, on September 23, 1971, to W. Phillip Deal and Kay Shinn Deal. She was graduated from Fort Mill High School in 1989 and, in 1993, she received her Bachelor of Arts degree in mathematics from Winthrop University. She continued her education at Winthrop and, in 1995, she received her Master of Science degree in mathematics. In 1996 she began her studies in the Statistics Department at the University of Florida.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.



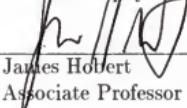
Andrew Rosalsky, Chairman  
Professor of Statistics

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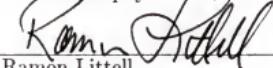
Malay Ghosh  
Distinguished Professor of Statistics

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James Hobert  
Associate Professor of Statistics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.



Ramón Litell  
Professor of Statistics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.



Irene Hueter  
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This dissertation was submitted to the Graduate Faculty of the Department of Statistics in the College of Liberal Arts and Sciences and to the Graduate School and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

May 2001

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Dean, Graduate School